THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE FOURTH YEARBOOK

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1929

SIGNIFICANT CHANGES AND TRENDS IN THE TEACHING OF MATHEMATICS THROUGHOUT THE WORLD SINCE 1910

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The National Council of Teachers of Mathematics is a national organization of mathematics teachers whose purpose is to

1. Create and maintain interest in the teaching of mathematics.
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3. Help the inexperienced teacher to become a good teacher.
4. Improve teachers in service by making them better teachers.
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Anyone interested in mathematics is eligible to membership upon payment of the annual dues of $2.00. This entitles one to receive the official journal of the National Council (*The Mathematics Teacher*). All correspondence relating to editorial and business matters should be addressed to *The Mathematics Teacher,* 525 West 120th Street, New York City.

EDITOR'S PREFACE

The custom of publishing yearbooks is rather generally established in several national organizations. The first three yearbooks of the National Council of Teachers of Mathematics have been well received. The first dealt with "A Survey of Progress in the Past Twenty-five Years," the second was devoted to a consideration of "Curriculum Problems in Teaching Mathematics," and the third dealt with "Selected Topics in the Teaching of Mathematics."

The fourth yearbook was made possible by the financial assistance of the American Commissioners of the first International Commission on the Teaching of Mathematics (1908-1922), Professors W. F. Osgood of Harvard University, David Eugene Smith of Teachers College, Columbia University, and J. W. A. Young of the University of Chicago. I wish to take this opportunity to express my own personal appreciation to these men for their assistance, without which the material for this yearbook could not have been secured.

I also wish to thank all my students at Teachers College who have helped in one way or another in the preparation of the last part of the yearbook relating to the United States. Especial credit is due Dr. Vera Sanford for translating the articles relating to Austria, Germany, Italy, Russia, and Switzerland, and to Dr. R. Lukavska-Stuerm for translating the article on Czechoslovakia.

It is hoped that the thoughtful consideration of the changes and trends in the teaching of mathematics in the leading countries of the world as herein presented will help to improve the teaching of mathematics wherever the yearbook is read.

W. D. Reeve

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SIGNIFICANT CHANGES AND TRENDS IN
THE TEACHING OF MATHEMATICS
THROUGHOUT THE WORLD
SINCE 1910

AUSTRIA

By DR. KONRAD FALK

Pädagogisches Institut, Vienna

Introduction. The following report is in two sections: the first deals with the mathematics of the *Volksschule*, the *Hauptschule*, and the continuation schools; the second, with that of the secondary schools.

The Austrian *Volksschulen* have courses of two, three, four, five, or six years. In the small towns in the lowlands and in the mountains, the pupils attend these schools from their sixth to their fourteenth years.[1] In the cities, the pupils attend the *Volksschule* from the sixth through the tenth year, going then to the *Hauptschule* from the ages of eleven to fourteen or else spending their tenth to their eighteenth years in one of the secondary schools where they are prepared for the university. Boys and girls who are obliged to go to work at the end of their fourteenth year are required to attend a continuation school for a period of from two to four years according to their trade. For example, a mechanic is obliged to take a three-year course, a hairdresser must train for four years, and a worker in the clothing trade two years. These continuation schools are in session from eight to twelve hours a week.

I. Volksschule, Hauptschule, and Continuation Schools

Arithmetic Curriculum. Prior to 1920 when the first plan of reform went into effect, the curriculum in arithmetic in the *Volksschule* was based on a very systematic plan. The following summary shows the general plan of the work:

First Year	Numbers to 20	Four fundamental operations
Second Year	Numbers to 100	Four fundamental operations and the simplest common fractions
Third Year	Numbers from 0.001 to 1,000	Computation with integers, common fractions, and decimals
Fourth Year	Numbers to 1,000,000	Computation with integers, common fractions, and decimals

[1] The period of compulsory education in Austria.—Editor.

Introductory work in numbers was constantly stressed and the four fundamental operations were studied and practiced.

Applied problems were given in each school year. The leading text was a four-book series by Krauss and Habernal [2] revised from the *Rechenbuch* of Močnik. In the days of the old Austrian Empire, this was translated into Italian, Czech, and Polish. Further details of the content and method of the work may be found in this text.

Geometry. The study of geometry began in the third school year in connection with the number work, and the subject was developed systematically, passing from lines to angles and surfaces and finally to solids. It was called the study of geometric form (*geometrische Formenlehre*).

Reform in Teaching Arithmetic. The reform in the teaching of arithmetic profited by the knowledge of the teaching of children gained about the turn of the last century. This reform calls for a thorough consideration of computation from many points of view, the linking of this with things within the child's experience or with things that may be brought into his experience; hence the use of the environment in the child's earliest number work and of materials drawn from his surroundings, his home, and his country in his later study. The actual fulfillment of these aims is made possible by the introduction of unified instruction in the *Volksschule,* that is, a plan of instruction which is a cross-section of the child's life and which shall develop many activities in combination with each other: speaking, counting, reading, expression in drawing and in writing, the meaning of number and space, developing these skills and practicing them in their natural sequence. Accordingly, there is no definite time schedule for these subjects of instruction in the first four years. This unified program requires the teacher to pay more attention to the needs of the child than was the case under the former curriculum, and the teacher is expected to select materials for work in arithmetic which shall be related to the environment of the children in the class. This type of instruction with its manifold pictures of life offers real situations of many types with which the work in arithmetic may be utilized without marring the unity of the projects themselves. Examples of the various classes can be seen in the following:

[2] These are the *Erstes, Zweites, Drittes, und Viertes Rechenbuch für allgemeine Volksschulen,* K. Krauss and M. Habernal. Osterreichischer Bundesverlag, Vienna and Leipzig.

SECOND CLASS
  Topic: *The First Snow*
    Reading—The White Street
    Study of Form—Ball, sphere
    Counting—Snowball fight
        Multiplication tables for 5 and 10

THIRD CLASS
  Topic: *The Sky and the Weather*
    The view from the window in the late evening
    Window panes. Square decimeters
    Cutting square decimeters from paper
    Measurement of surfaces such as a desk or table by applying squares of paper to them, thus giving an introduction to the measurement of rectangular surfaces

FOURTH CLASS
  Topic: *A Trip into the Wachau* (the most beautiful section of the Danube valley)
    Arithmetic—Reading the map of Austria, computing mileage, computing the number of vines (the Wachau is a wine-producing district)
    Theoretical work—Multiplication with two-place numbers

These illustrations give an insight into the mode and manner of the instruction in arithmetic in the *Volksschule*.

The work differs in certain respects from that given formerly. Fractions and the writing of fractions are now first introduced in the third class and the work is limited to the unit fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{10}$. In the fourth class the following fractions are introduced: $\frac{1}{10}$ and $\frac{1}{5}$; $\frac{1}{10}$ and $\frac{1}{100}$; $\frac{1}{6}$ and $\frac{1}{3}$. The order of operations (integral numbers) is division, multiplication, addition, and subtraction. These operations are illustrated by cutting and combining sectors from circles made of paper. Drawings are made later. Decimal fractions are introduced in the fourth class in connection with problems taken from the field of business. The first ones used are the hundredths as 3.25*m*, 4.25m^2, 4.27*q*, 3.36*s*, 4.12*hl*. Later tenths and thousandths are introduced. In all of this work, the teachers are expected to bring the subject matter close to the children's interests through the use of real materials, and they are also expected to develop the work in arithmetic from these same materials.

Upper Classes. In the upper classes of the *Volksschule* (ages 10 to 14) and the *Hauptschule* (ages 10 to 14), the subject matter is presented in a fashion that is more and more systematic but nevertheless related to the child's life. For example, the study of the Danube as a means of transportation brings with it the ideas

of kilometers, of addition and subtraction of three-place numbers and of higher numbers.

The plan of study for the introductory lessons is given in the syllabus published in 1928.[3]

Instruction in Geometry. Instruction in geometry begins in the first school year, for at this time the aspects of space are clearer to the child and richer and more permanent than are his concepts of numbers. The work of the first school year is called the cultivation of space perception (*Pflege der Raumanschauung*). Certain concepts from geometry fundamental in the work which begins in the child's eleventh year are presented in his fourth school year (that is, when he is 9 to 10 years old).[4]

Instruction in Arithmetic. In the continuation schools, instruction in arithmetic and in the study of space is entirely dependent upon the student's trade, for the work is based on the mathematical experiences needed for the trade. Accordingly, those working in electrical trades learn how to reckon with electrical quantities. Their lessons include calculations of the strength of a current, plans for installing lights, laying cables, and the like. In addition to this, they are instructed in simple bookkeeping. Before 1907 this instruction, which is now entirely devoted to practical things, bore the name of the trade for which the students were preparing, namely, shopkeeping course (*gewerblichkaufmännischer Unterricht*). As a general rule, special collections of problems dealing with the different crafts were given to the pupils in the form of a business story. Since 1920, the needs of the student's environment have been considered also—the apprentice at home (budgets), the cost of illness, alcohol as a poison, and other problems.

In all types of schools, it is required that rules and theorems be developed by the class as a joint piece of work and that these be expressed in the pupils' own words. For this reason, the newest textbooks in arithmetic give either few rules or else none at all.

II. Secondary Schools and Training Colleges

Secondary Schools in 1910. In 1910, the secondary schools in Austria were humanistic *Gymnasien* with an eight-year course, *Realschulen* with a seven-year course, or *Realgymnasien* with an

[3] *Lehrplan für die erste bis fünfte Schulstufe der allgemeinen Volksschulen nebst den dazugehörigen Erlauterungen.* Vienna and Leipzig, 1928.

[4] See Konrad Falk, *Die Pflege der Raumanschauung in der Grundschule.* Vienna, 1922.

eight-year course. In general, the children were admitted to these schools at the age of ten on the basis of a competitive examination, and on completing the course they went on to the university, or to an institute of technology.

Mathematics Curriculum Since 1910. Since 1910, the mathematics curriculum in these schools has had to comply with the government regulations of 1909 and it has sought to realize the aims that are there set forth. It is required that the foundations of the function idea be laid in the second class with the consideration of simple cases of functional thinking as illustrated in the understanding of the changes in the shape and size of geometric figures with changes in their dimensions. In connection with functional thinking in the fourth class comes the study of the variation in the result of computation with the changes in the numbers involved as applied especially to the handling of equations. Associated with this also is the graphic representation of linear functions and its applications to the solution of equations of the first degree. In the higher classes, quadratic, exponential, and logarithmic functions, as well as trigonometric functions, are investigated graphically. The preliminary work is so arranged that the elements of the differential and integral calculus may be introduced into the curriculum and so arranged also that certain work in sections of mathematics and physics may be discussed more comprehensively than heretofore.

In its details, the plan of study indicates an ordered sequence suited to the intellectual development of the student. It seeks to concentrate the pupil's interest in arithmetic and geometry and to establish closer bonds between this work and other branches of study than was the case before. The student is also to be especially informed through manual activity of the great rôle played by spatial perceptions.

Thus, the method of instruction had to be changed, both in the exposition by the teachers on the one hand, and in the practice and experience of the student on the other. The amount of formal instruction by the teacher had to give way before the discussion of the students in which they share in question and answer. Formal tests are used only when the teacher lacks a clear picture of the pupil's performance or when that picture is unfavorable.

The School Reform of 1918. The school reform of 1918 was begun while we were in the midst of bringing certain changes to

pass. Most notable of the changes is the organization of the first four years of the secondary schools known as the *Deutsche Mittelschule*. Following this comes a four-year *Oberschule*. This may be of four types: classical, modern languages, mathematics-science, or the *Deutsche Oberschule* with English as the one foreign language. At the same time, the Vienna investigators thought it of importance to introduce the plan of study of the *Deutsche Mittelschule* into the numerous schools which are compulsory for children from the ages of eleven to fourteen—an important step in the unification of the general school program.

Both the plan of study and the method of instruction in mathematics have undergone further improvements. These have been in the direction of a more thorough working out of the aims established in 1909. The development of special concepts has been even more strongly emphasized, and the teachers are directed to make close connections between arithmetic and geometry. Functional thinking has become more important. Above all else, the instruction in the secondary schools has been influenced by the reforms in teaching now taking place in the *Volksschule*. Vocational instruction is being introduced for the first time. The teacher is not permitted to supply the pupil with information which he can gain for himself by the use of his own hands or his own mind. The teacher may stand beside the pupil only to guide him. It has actually been shown that students are able to work out almost the whole content of the syllabus independently and that it is only necessary for the teacher to suggest certain fundamental ideas. Among these concepts are the idea of a system of coördinates, the idea of a goniometric function, the idea of difference quotients—discoveries which mankind itself has had handed down in a long period of time from its greatest men of genius.

It has proved necessary also to avoid following a standard procedure in the solution of problems, since each method discovered by the pupil is of value for him, and the more methods he discovers the better. A comparison of these methods with regard to their simplicity, suitability, and economy of time gives deeper understanding and leads to more efficient procedure in the solving of problems. Warning is given against too early introduction of the abstract, against mechanical drill on rules, and against the use of definitions in set form. The student must be given concrete problems as long as possible, he must see with his own eyes, measure, cut,

fold, construct, and draw; he will then himself develop the rules and find terms to express them; he will have a firmer grasp of these rules, and he will make more efficient use of them.

Consequently, collections of problems occupy the chief place in the more recent books intended for mathematical instruction. Number material, and representations of objects from daily life are introduced and the customary material of the formal textbook is included only in an appendix, called *Leitfaden.* In some cases the formulation and transcribing of principles is entirely the work of the class or of the individual student.

The Mathematical-Scientific Oberschule. The curriculum of the mathematical-scientific *Oberschule* employs in mathematics the materials used in the former *Oberrealschule,* but the course is divided into four years instead of three. In the other types of *Oberschulen* spherical trigonometry is omitted and the amount of analytical geometry and calculus is somewhat reduced. Premature development of a scientific system is avoided; a comprehensive survey and review is reserved for the final work of the highest class where axioms may be considered and where a sound scientific structure may be built up.

The experiments mentioned above were made between 1920 and 1927, working up from the lower classes. The results promise further steps in advance. The school law of 1927 assured the centralized administration of all the *Untermittelschulen* with the *Hauptschulen,* the four classes of the *Gymnasien* being a notable exception; and the *Realschule* was developed into an eight-year type. The plans of study published in 1928 by the Minister of Education have as the chief foundation in mathematics the experimental plans of the *Deutsche Mittelschulen* which have been cited, and the above-mentioned *Oberschulen.* Notes on method to accompany the plan of study are announced but have not yet appeared.

Teacher Training. The training schools for teachers in the *Volksschulen* provided a four-year course culminating in a final examination for their pupils from about 15 to 18 years of age. Prior to 1923 these had a mathematics curriculum composed of the subject matter of the *Untermittelschulen.* Since that time, at the instigation of the institutions themselves, the course has been extended, so that now the subject matter of the *Obergymnasium* is required for study. The principles in method are naturally like those of the *Mittelschulen* and are in this case of even greater importance, as the

pupil in these schools is following out for himself the fundamental principles which he is later to apply. A *Pädagogisches Institut* established in Vienna is available for teachers in *Volks-* and *Hauptschulen* for their training and further advancement. The course in mathematics in the *Institut* offers a review and a more thorough study of the material already known, a further insight into the study of the properties of number equations, higher equations, methods of approximation, functions, and series. The calculus is taken up more thoroughly and in greater detail, and seminar exercises offer discussion of scientific questioning and practical teaching problems.

CZECHOSLOVAKIA

By DR. QUIDO VETTER

Prague

Introduction. In order not to extend the length to which this article is limited, we shall confine ourselves to the mathematical subjects, arithmetic, geometry, and descriptive geometry, as taught in the secondary schools proper, *Gymnasien* and *Realschulen*, because these schools determine also the character of instruction in the professional schools whose particular specialization results from their narrower aim. Mathematical instruction in the universities will be dealt with only so far as these prepare the future professors of the secondary schools.

Two Periods under Discussion. For the past twenty years, Czechoslovakian life, as well as the development of the Czechoslovakian schools, has been divided into two basically different periods. The dividing line was the twenty-eighth of October, 1918, when the independence of our state was regained. The first period was Austrian and falls again into two parts, one before the war and the other during the war, after 1914. The part before the war is characterized by the Austrian reform of secondary schools in 1909, with the collaboration of the university professor, Dr. Fr. Drtina, who later organized the Czechoslovakian Ministry of Education. This reform brought relief to the overtaxed pupils of the secondary schools, an intensified school activity, and a preparation for an "active" school, introducing, for example, so-called examining for orientation. A marked change was then introduced into the teaching of mathematics; in response to the reform movement under the leadership of Professor Felix Klein, the conception of the function was made the center of the teaching of mathematics, and the infinitesimal calculus was introduced.

In Austria and also in Czechoslovakia almost all the secondary schools are state schools. The private schools are bound by the same rules as the state schools, especially as to the curriculum and qualifications of the teachers, and as to textbooks which the Ministry of Education has to approve. Therefore, the outside con-

ditions of teaching mathematics within the boundaries of the present-day Czechoslovakian Republic did not differ up to the year 1918 from those of the other lands of the old monarchy. For that reason the report of Dr. K. Vorovka, Dr. L. Cervenka, and Dr. V. Posejpal in the year 1914 [1] limited itself only to the Czech mathematical textbooks and those of related fields, because they represented the main sphere in which the Czech characteristics could express themselves. It would be superfluous to discuss here that period, which will certainly be described more thoroughly in other similar reports.

The New Movements of Prewar Times. I shall, therefore, only point out that prewar times were characterized by Czech teachers of mathematical subjects working themselves into the new methods and the new teaching material prescribed by the new curricula, gathering experience in their use, and beginning to criticize the situation, using these new curricula as a basis. The teaching of mathematics in Czechoslovakia was helped by the fact that simultaneously with the Austrian reform of 1909 the generation born around 1880 came into action. This generation included several intelligent persons full of ideas concerning the reform movement already mentioned. Their leaders included those mentioned above. They are Dr. B. Bydzovsky, professor in the Charles University at Prague; L. Cervenka, government counselor and land school inspector in Prague; Dr. J. Jenista, ministerial counselor and chief of the pedagogical department of the Ministry of Education, lately deceased; Professor Dr. B. Masek, vice-director of the state observatory; J. Muk, *Gymnasial* [2] professor in Prague; J. Pithart, *Realschule* director in Prague; Dr. V. Posejpal, professor in Charles University in Prague; K. Rasin, *Realschule* professor in Prague; Dr. J. Seyfert, professor of Masaryk University in Brne; Dr. M. Valouch, former section chief of the Ministry of Education; and Dr. K. Vorovka, professor in Charles University at Prague.

Results of the Great War. The war brought stagnation to all public life, and so to the schools. Lack of teachers resulting from the draft, fear for the future of the nation, economic hardships, and lack of food exhausted the teachers left at home, so that there

[1] *Die Lehrbücher für Matematik, Darstellende Geometrie und Physik an den Mittelschulen mit böhmischer Unterrichtsprache, Berichte über den matematischen Unterricht in Oesterreich,* Heft. 13.

[2] The *Gymnasium* is the classical high school and the *Realschule* is the more modern type.

was, at first, no chance for more spirited action. But when they realized in 1917 that the Austrian government was preparing a reform of the secondary schools directed against the oppressed nations, the official Czech representatives formed a committee for the reform of the secondary schools. Its official character held their discussions to the general features and made them consider the current conditions.

Union of Czech Mathematicians and Physicists. Besides this official commission there was the "Union of Czech mathematicians and physicists," in which are organized all the Czech teachers of mathematics and physics of the secondary schools as well as the universities, and which is one of the first professional Czech associations to prepare and work out a detailed proposal for a new Czech secondary school.

In the summer of 1917 the board of the Union selected a committee for reforming the secondary school which divided itself into committees for mathematics, descriptive geometry, and physics. The first two authors of the above-mentioned report and textbooks were joined by Dr. B. Hostinsky, professor in Masaryk University at Brne; Dr. J. Kounovsky, professor in the Institute of Technology at Prague; Dr. B. Salomon, professor in Charles University at Prague; and the writer. At the outset, they decided as a result of a motion made by Professor Vorovka to discuss the free Czech school as they conceived it, regardless of existing conditions, which meant, of course, the school in the future Czechoslovakian State.

The Proposal of the Union. We still have on the one hand the classical *Gymnasium* with Latin and Greek, and on the other the *Realschule* with modern languages and more thorough instruction in mathematics and the natural sciences, while the two types of the *Realgymnasium* with Latin, modern languages, and descriptive geometry, together with the so-called Decin type *Gymnasium,* stand between. The proposal of the Union aims at postponing the decision about the direction of the studies as far as possible until the pupil is at a more mature age. It suggests a common four-year basis on the native language and civics, during which period all pupils would learn the fundamentals of drafting. In the two following years Latin in one division would alternate with descriptive geometry and drawing in the other. The seventh form would split into two departments: those of *Gymnasium* and of the *Realschule,*

while the eighth form would have two divisions: philological-historical and mathematical-technical.

Curricula for Mathematics and Descriptive Geometry. The committees worked out detailed curricula for mathematics and descriptive geometry. For the first form four instead of the customary three hours a week of mathematics are suggested; in geometry is added the remainder of the work on axes and planes. For forms 2 to 6 the Union offers, with the exception of minor changes, the earlier curriculum of the *Gymnasium* up to the work with four-place logarithmic tables, and compound interest calculations, as well as the fundamentals of plane trigonometry. The necessary double level of teaching is intended to be interpreted in such a way that material taught in the lower grades may be summarized and deepened in the higher ones, so that the detailed teaching of the whole field will not be repeated. In the seventh form of the *Gymnasium,* combinations, the binomial theorem for positive integral powers, the fundamentals of probability, and insurance mathematics, as well as the additional plane trigonometry, are included. With these the teaching of mathematics is to end, and from the present offering analytical geometry is to be dropped. The latter would, however, figure in the *Realschule* division, where it would be taught up to the work on the circle. For the eighth form of the natural science division binomial equations, geometrical representation of complex numbers, De Moivre's formula, and the fundamentals of infinitesimal calculus are suggested. For the eighth form of the mathematical-technical division numerical and graphic solutions of higher equations, analytical geometry of conic sections, and the completion of planimetry are added. In descriptive geometry the present material of the *Realschule* is to be taught up to the seventh form; in the eighth form of the mathematical-technical division the fundamentals of projective geometry and conic sections, and the theorems of Pascal and Brianchon are to be taught, while the material completed in the lower grades is to be deepened with the alternative use of suitable types of projection, and with the explanations of the methods of representation in constant view.

Aim of the Secondary School. At the same time the Union took a decisive negative stand against the attitude of the professors of the university technical schools who suggested the shifting of a part of the theoretical teaching material of their schools, especially that of mathematics, descriptive geometry, and physics,

to the secondary school. The Union held the opinion that the aim of the secondary school is mainly that of general education and not that of professional preparation.

I have dealt with these reform proposals in detail, because they have influenced deeply the development of this question in our country and will undoubtedly continue to do so.

Work of the Ministry of Education. Soon after the bloodless revolution of the twenty-eighth of October, 1918, the Union came out with its proposals. This stimulated further discussion in the committee of the Ministry of Education, whose chairman is Professor Dr. Bydzovsky. This committee brought forward another proposal which in certain respects is similar to that of the Union and which now forms the basis for a study by able specialists and will doubtless become a foundation for future changes. For the present the belief has become prevalent that a sudden change would not be beneficial to the secondary school; a slow development with a gradual realization of the reforms to be sought is considered much healthier.

In accordance with this proposal and under the influence of the proposal of the Union, as early as the year 1919 the number of periods of religion and classical languages was somewhat limited in the schedule, the hours of the native language were increased, and in the first form of all the secondary schools the teaching of mathematics was increased from three to four hours a week; in *Realschulen* and *Realgymnasien* the teaching of descriptive geometry was increased by one period a week, and in *Realgymnasien* the latter was shifted into the last two or three grades of that type of school.

Only in Slovakia, where under the Hungarian rule there were no Slovak secondary schools whatever and the primary schools were allowed only as private schools with schedules overtaxed by the teaching of the Hungarian language at the expense of all the other subjects, was it impossible to change the schedule in favor of the subjects of mathematics and the natural sciences. These differences were at last annulled by the decree of the Ministry of Education of June 7, 1927, by which, with small variations resulting from local conditions (differences in the number of periods of religion and modern languages), the division of the schedule for the individual subjects is equal for all the parts of the Czechoslovakian Republic.

Weekly Mathematics Schedule. Even in the schools with different languages of instruction (Czechoslovakian, German, Hungarian, Polish, or Russian), the schedule is practically the same. For the teaching of mathematics the following weekly schedule is in force at present:

School	I	II	III	IV	V	VI	VII	VIII	Total
Gymnasium	4	3	3	3	3	3	3	2	24
Realgymnasium	4	3	3	3	3	3	3	2	24
Reformed Realgymnasium	4	3	3	4	3	3	3	2	25
Realschule	4	3	3	4	4	4	5	..	27

For descriptive geometry and elementary drafting, the following schedule is in force:

School	I	II	III	IV	V	VI	VII	VIII	Total
Realgymnasium	..	..	..	..	..	..	2	2	4
Reformed Realgymnasium	..	2	2	3	2	3	..	..	12
Realschule	..	2	2	3	3	3	2	..	15

Also the Decin type *Gymnasium*, i.e., higher *Realgymnasium* with three divisions of its higher studies on top of the identical four-year basis with Latin, was rearranged as follows:

School	I	II	III	IV	V	VI	VII	VIII	Total
Mathematics									
Gymnasium	4	3	3	3	3	3	3	2	24
Realgymnasium ...	4	3	3	3	3	3	3	2	24
Realschule	4	3	3	3	4	4	4	4	29
Descriptive Geometry and Drafting									
Realgymnasium ...	..	..	2	2	..	..	2	2	8
Realschule	..	..	2	2	2	3	2	3	14

Shortage of Teachers. The activity of the teachers of the secondary schools of all the departments was being exhausted, in the first ten years after the formation of the republic, by the increased tasks of actual instruction. A shortage of teachers resulted from the appointment of many to the newly organized Ministry, and into the newly established schools in Slovakia and Sub-

carpathian Ruthenia. As has been said, under the Hungarian rule there were no schools in Slovakia and Subcarpathian Ruthenia (which have together 3,603,148 inhabitants) with Czechoslovakian or Russian language of instruction, though the great majority of the population is Czechoslovakian (2,018,550, according to the census of 1921) and Russian (458,145). Within a few years after 1918, fifty-four Czechoslovakian and sixteen Russian secondary and normal schools were established there. It is not necessary to add that this lack of teaching personnel, which is only now being relieved, affected also the teaching of mathematical subjects.

Textbooks Published by the Union. The changes in the teaching material show best in the changes of the most widely used textbooks, published by the Union. The first edition is described in detail in the report of 1912. In an arithmetic by Cervenka, the examples were adjusted to the new conditions, the present unit of currency (the Czechoslovakian crown) being taken everywhere into consideration, and sometimes even the prices in the examples were changed according to current prices. It is certainly to be recommended that similar changes be made in all the examples in the next editions. The parts on political economy were simplified. Although in the first edition they were very good, they sometimes overtaxed the interest and comprehension of eleven- and twelve-year-old children. On the other hand, there was included ratio and proportion, as prescribed by the changed curricula, and also devices for convenient and quick calculation (addition, multiplication, and squares). There is, also, an instructive chapter on the differences between our symbols and those used in foreign, especially Anglo-Saxon, countries. The geometry by Valouch remained almost untouched.

The arithmetic by Bydzovsky, however, underwent a much greater change. It was criticized for having overstressed graphic delineation. In the last edition some parts of the graphic delineation were omitted, for example, a very instructive explanation based on a graph when in an equation of the second degree one root becomes infinitely large; also, the graphic drawings, which in the first edition permeated the whole textbook from the very beginning of the first form, were omitted in about the first quarter of the book, and combined into a special chapter falling between the materials for the fourth and fifth forms. These changes will not be generally pleasing to the friends of graphic representation in our instruction.

Finally, everything which was unnecessary was omitted or the important material was simplified in order to prevent overtaxing of the pupils.

Because of financial reasons, examples which are not part of the explanation were referred into a separate Summary of Problems. The order of the teaching material was also slightly changed.

A similar situation exists with reference to Vojtech's geometry for higher forms. There inversion was omitted, the parts on the higher curves were somewhat simplified, as well as the graphic solutions of numerical equations, and the infinitesimal calculus; in the edition, however, for *Gymnasien* there was added some work on projection. The style is more concise and the problems were referred for financial reasons to the Summary of Problems.

The descriptive geometry by Pithart and Seyfert remained untouched. As to the other textbooks dealt with in the above-mentioned report, we have only to add that the textbook by Klira and Rasin was worked over in the year 1925 by B. Matas and that there is a new textbook in arithmetic for the higher forms by J. Muk, all of which are popular. In Muk's textbook, where proper attention is paid to graphic delineation, post-war conditions are expressed in the part on compound interest by explanations of inflation, deflation, and devaluation of currency.

German secondary schools of the Czechoslovakian Republic use almost entirely the old Austrian textbooks for mathematics which were only recently submitted to the Czechoslovakian Ministry of Education for a new approval. The only new textbook is the descriptive geometry in four volumes by A. Schwefel, *Deutsche Realgymnasium* professor at Prague. It differs from the Czech textbooks by a larger quantity of material. Especially in the fourth volume, there are parts which are not included in the curriculum; for example, shadows of rotation planes, straight-line angles, angles of planes in central projection, and the treatment of surfaces in perspective.

Methods of Teaching. The method of teaching in the secondary schools of old-time Austria was determined by the Instructions of 1899, and the decree of the Ministry of Education of 1909, partly in amendments to the curricula, partly in the rules about examinations and classification. These decrees stressed the heuristic method, already in use, as well as the coöperation of the whole class during instruction.

Dr. Cervenka's Lecture at Prague. As to the teaching of mathematics and descriptive geometry in the Czech schools of the Czechoslovakian Republic, and especially those in its largest section, Bohemia, the lecture of Land Inspector Lad. Cervenka at Prague is instructive. The lecture was delivered on April 4, 1925, at the conference of Prague professors of mathematics and descriptive geometry. We cannot go into the details which set up for the teaching of our subjects well worked-cut modern aims, frequently dealing with detailed problems. We shall mention only a few of the leading ideas.

Dr. Cervenka, the chief of the teaching of mathematics in the secondary schools of Bohemia, considers the aim in teaching mathematics to be the acquiring of positive knowledge, as well as general formal education. He stresses the development not only of the intellect, but also of the educational elements in the ethical sense, in the development of the emotions and will. Because in our subject more than in others the pupil's achievement depends on his having mastered the material, the didactic responsibility of the teachers of mathematics is so much the greater. Consequently, Dr. Cervenka demands from the teacher, to whom a considerable freedom in methods is left, a thorough didactic and methodical preparation, not only on the whole year's material, but also on each teaching period, during which there is, of course, also necessary an emphasis upon the detailed situations in the classroom. He, therefore, calls for a thorough university preparation in the special didactics of mathematical subjects on the part of future professors of the secondary schools and, emphasizing the present arrangements, he demands a further possibility of education for the teacher and insists that he be not overtaxed by too many hours of instruction. He also recommends mutual classroom visits.

Even Dr. Cervenka considers the heuristic method best and the only possible one in the lower grades, with the constant coöperation of the whole class. However, he warns against its extremes. He advises the teacher in the higher grades to introduce from time to time an uninterrupted model explanation, proof, or solution. He recommends that the pupils themselves be given opportunity for such uninterrupted work, perhaps even to a coherent explanation of the material, prepared on the basis of the references furnished by the teacher. In all the grades he thinks that the independence of the pupil should be considered. This, he says, is easily

ruined by the formal question method. He believes that the pupil's activity should be encouraged by abundant practical measuring, weighing, and the like, as well as by the pupil's choice of the examples, as in the case of subjects for drafting. He stressed also the importance of making the instruction interesting, especially by the introduction of historical comments, and pointed out the educational value of the biographical element, from the lives of the great mathematicians. He emphasized the need of sharpening the judgment by oral examples taken from the practical life of the pupil's environment.

From the details I mention only the demand for an exact and uniform terminology, for uniform arrangement of solutions and proofs, for a certain minimal canon of formulas to be memorized, for an abundant practice of numerical calculation in all the grades to be subsequently memorized, and for a consideration of the speed of a pupil's work in the various topics in the field of mathematics in connection with which the American tests were called to attention.

The Activity School. The teachers of the secondary schools stress lately, also, the methods of the activity school. The reports of the pedagogical section of the sixth convention of the Czecho-slovakian natural scientists, physicians, and engineers, in which also the mathematicians are taking part, deal with this problem, and we hope that they will give impetus to the further development of these methods of teaching in our subjects.

Problems of Teaching Mathematics. For a clear discussion of the problems of teaching mathematics the "Didactic-methodical Appendix" of the *Journal for the Practice of Mathematics and Physics* is also important. It is published by the Union and appears for the third year in 1927-28 under the editorship of J. Fridrich, *Realschule* professor at Prague.

In their individual study of mathematical and physical sciences outside of school, the pupils of the secondary schools are stimulated by the *Outlook of Mathematics and Natural Science,* edited by Professor Dr. V. Rysavy. An approximate picture of present-day teaching of mathematical subjects is offered by the statistics of classification of twenty-nine Prague secondary schools which were worked out for the convention by Professor J. Muk. The final results in percentages are given in the table on the following page:

SUBJECT	RECORD				
	Very Good	Good	Passed	Unclassified	Failure
Mathematics	13.4	31.0	49.9	0.3	5.4
Descriptive Geometry.	17.9	35.2	43.6	0.4	2.9

The study of future Czech professors of the secondary schools is better provided for in our state than in old Austria-Hungary. At Masaryk University in Brne there was established two chairs of mathematics; lectures on analytics were given by the well-known Dr. M. Lerch, and after his death by Professor Dr. E. Cech; lectures on geometry are given by Professor Dr. L. Seyfert of Charles University at Prague and the Czech Institute of Technology at Prague, where the future professors of descriptive geometry spend two years before their university studies. As a result all are much better equipped than before. In the Czech Institute of Technology there are six professors of mathematics, Dr. V. Hruska, Dr. J. Kloboucek, Dr. F. Radl, Dr. K. Rychlik, Dr. J. Svoboda, and Dr. J. Vojtech, and three associate professors, Dr. K. Dusl, Dr. V. Hlavaty, Dr. B. Machytka. Three professors, Dr. F. Kaderavek, Dr. J. Kounovsky, and Dr. V. Hruska, and one associate professor, Eng. B. Chalupnicek, teach descriptive geometry. Lectures are given to candidates of the teaching profession on the review of the history of mathematics by the writer of this article.

In Charles University, under Austrian rule, there were only three professors of mathematics, Dr. J. Sobotka for geometry, Dr. V. Laska for applied mathematics, and Dr. K. Peter for analytics. Now they have been joined by Dr. K. Bydzovsky for geometry, Dr. M. Koessler for analytics, Dr. E. Schoenbaum for insurance mathematics; and by four assistant professors, Dr. K. Rychlik for algebra, Dr. V. Jarnik for analytics, Dr. B. Machytka for geometry, and the writer.

In addition, Dr. K. Vorovka gives lectures on the philosophy of mathematics. The writer, as assistant professor with the title of university professor, lectures on the history of mathematics. He also has a lectureship on didactics and methods of mathematics: one year he lectures on the general didactics of mathematics, in another year on special methods of one of the branches—arithmetic, geometry, or descriptive geometry. The students themselves report

on new methods or perform some experiment with the teaching material of the secondary schools. Dr. Cervenka strongly favors this arrangement, and with his help and advice, as well as the permission of the Ministry, the writer organized the classroom visits of the students and their practice in the didactics of mathematics at three of the Prague *Realschulen.* After each visit the students report and discuss their experiences. The classroom visits have proved very successful and they should be continued and developed further.

Except for the Ministry's approval, the organization has a private character, and should be taken care of by the Ministry itself, after an agreement with the directors in question and individual well-known professors of the secondary schools. The original permission of the Ministry should hold for classroom visits in all the secondary schools of the university town, out of which certain exceptionally good teachers should be chosen, according to the suggestion of the land inspector, in order to take charge of this difficult and responsible task. Of course, they would have to be recompensed for their special work. In addition, the possibility of visiting should be afforded to a larger number of the students at the same time; they should observe several successive periods of mathematics and descriptive geometry in the same class, and after visiting is over the teaching professor himself, ordinarily in the presence of the university instructor of didactics and methods of mathematics, should hold a longer discussion with the students about the performed teaching which they witnessed and on which they have to hand in a report. Further, it would be necessary to establish, aside from the two hours a week of the lecturing on didactics of mathematics, another two hours of seminar work. It would be advisable to include the didactics and methods of mathematics and descriptive geometry among the subjects of the state teachers' examination for the candidates of the professorship of these subjects. Finally, the university should offer also regular lectures on elementary mathematics from a higher point of view.

These arrangements could be brought about sooner and more easily than the materialization of the proposal of the committee on the reform of the secondary school, in the Ministry of Education, which suggests an establishment of special model secondary schools, to which the candidates would be assigned after the completion of their university studies, and where their "methodical-didactic" preparation would be centered. This proposal is still under debate.

ENGLAND

By G. St. L. Carson[1]

Introduction. Many of the changes in England since 1910 are due to or are connected with the increasing systematization of public education. To give a comprehensive account of what has happened it is necessary, therefore, to state, in outline at least, those main features of this systematization which have affected the teaching of mathematics. Up to the present time they have been almost entirely concerned with the state-aided secondary schools, that is, schools which receive their pupils at 11 or 12 years of age and retain them until 16 at least;[2] but, as will be seen later, a problem of a different type has arisen and is rapidly becoming an urgent matter.

We shall discuss these changes in turn, pointing out as many of the details as possible before giving problems that confront us at the present time.

I. First Marked Change

Admission of Pupils. The first marked change in these secondary schools is in the method of admission of the pupils. The demand for places having become much greater than the supply, the schools have been able to impose their own terms for admission, and these, broadly speaking, have been two in number. The first is that the pupil on entry shall not exceed a certain age which at first was often 13 years but is now rapidly tending to become 12 years; the second, that he must possess some reasonable minimum equipment in English and arithmetic. In consequence, the classes are more homogeneous as regards both the ages and the minimum attainments of the pupils, and there is a definite minimum course of mathematical instruction which can be planned for the school as a whole, a condition of things by no means common formerly and not always found even now.

[1] This report has been written with the consent of the Board of Education, by Mr. G. St. L. Carson, M.A., H.M. Inspector of Schools and Staff Inspector in Mathematics. It should be clearly understood that the opinions expressed are his own, and that they do not commit the Board in any way.—Editor.

[2] Approximately Grades 7-11 of the schools of the United States.—Editor.

II. Second Development

Length and Extent of the Course. In considering the length of this course, and the extent to which it is completed by all pupils, we come to the second development in secondary schools. It is now generally recognized that education of this type will inevitably fail to be effective unless continued for four or five years, and in consequence pupils have tended, in increasing numbers, to stay at school until they have actually or nearly attained the age of 16 years. It is true that this stay is not always voluntary in that parents are often required, when children enter a secondary school, to sign a corresponding undertaking; but it is finding acceptance as a cardinal feature of secondary education and the schools are benefiting correspondingly. The type of pupil, formerly very common, who goes to such a school for a year or two "to finish" is now almost unknown.

Apart from the abler pupils who stay longer for more advanced work and who will be considered later in this Report, a secondary school can, therefore, be regarded as receiving its pupils, under a minimum qualification, at the age of 11 or 12 and retaining them for four or five years. In consequence, classes are much less often subdivided into groups of which the worst is regarded as practically hopeless, and teachers have been faced, more than ever before, with the problem of instructing a whole school, class by class, in a common mathematical syllabus. The effect on the quality of the instruction itself has been striking, for the teachers have been led to attempt many more methods of presentation instead of keeping to those which were traditional and passing over pupils to whom these particular methods did not appeal; and to find these new methods the teachers have used their own ingenuity and have turned for help and inspiration in striking numbers to meetings of the Mathematical Association and to courses on the teaching of mathematics. For this a very large amount of credit is due to the teachers themselves; but to view the matter aright it must be remembered that what these teachers have in fact done is not so much to initiate a change as to respond—admirably, as has just been said—to a need brought about by legislative and administrative action.

III. Another Development

First Examination. Illustrations of this general statement are of course necessary, but they must be preceded by an account of

another development; namely, the institution of an examination, called the "First Examination," appropriate to the termination of a normal secondary school course. All that need be said here is that certain existing examinations, which were more or less suited to this purpose, have, by concerted action, been shaped so as better to suit it and are now taken, not by selected pupils, but by whole classes. In its general aspects this change is still a matter of controversy; but in regard to mathematics the Examining Bodies, with the aid of teachers, have devised tests which meet with fairly general approval except for the residue of pupils, mostly girls, who are said to be incapable of learning the subject. This fact is important because the syllabi and papers supply a statement of what is at present found to be possible for most of the pupils; and, as will shortly be seen, in certain respects the contents of these syllabi provide conclusive evidence of progress.

Normal Minimum Curriculum. Turning now to the normal minimum curriculum; it includes arithmetic, algebra, geometry, and in almost all cases trigonometry of a simple numerical kind; and these are the subjects found in the First Examination syllabi, trigonometry being sometimes optional and sometimes absent. Arithmetic now includes logarithms; algebra includes graphs but goes no further than the traditional quadratics and progressions. Geometry usually includes Euclid's first six books; and trigonometry includes only the simplest formulas and identities. This being the present usual minimum syllabus, there remains to be considered to what extent teaching and achievement have progressed. The several branches of the subject will be considered in turn.

The Arithmetic Syllabus. The only marked change in the arithmetic syllabus is the introduction of logarithms, usually in the second or third year of the course, for practically all pupils instead of a chosen few. In part, this was due to the needs of teachers of science; but it would probably have happened, sooner or later, in any case, on account of the desire of the mathematics teachers to give their pupils more command over problems of varied types. However that may be, the work is good and has given the pupils an added source of power, and therefore an increased interest.

Technique of Teaching. In the technique of arithmetic teaching there is no such marked change: probably none is to be expected. But there is a markedly more critical attitude among the teachers themselves, as is well illustrated by their treatment of the

multiplication and division of decimals. For some years the method known as standardization was powerfully advocated and widely used in virtue merely of that fact; but there is now a growing disposition to judge this and other such matters at first hand and in the light of experience—that is, the corporate body is finding a mind of its own.

Achievement in Arithmetic. As to achievement in arithmetic, the efforts of the teachers have undoubtedly led to increased power and interest. Problems which would formerly have been considered hard are now found easy, and this is due to better teaching. But it is widely felt by teachers themselves, as by others, that standards of speed and accuracy in simple computations are not what they should be, and methods for raising these standards are being more and more considered.

The Algebra Syllabus. In the algebra syllabus, there has been little change: it has still the traditional content, as has already been said. It is of interest, however, to observe that the attempt, initiated some twenty-five years ago for example, to limit the scope of the formal operations to fractions having single terms for their denominators, has failed, apparently through the tacit opposition of teachers themselves. Rightly or wrongly, it seems to be felt that something like the present syllabus is the least that is worth having at all. Many of the more progressive teachers have not always taken this view; and, no doubt, some do not now; but on the other hand, there are now many who have been convinced by experience that anything less than the present syllabus would provide a tool of little use.

Technique of Teaching and Achievement. In the technique of teaching and in the attainment of pupils there seems to have been little change beyond that which has followed from the efforts already described of the teachers themselves. In short, apart from the tacit assertion of the minimum syllabus and the general improvement in teaching, algebra appears to have progressed but little. The truth is probably that it needs an outlet such as logarithms provided for the work in arithmetic. Compared with the size of the tool, the pupils have made, as yet, but little use of it.

Geometry. In geometry, on the other hand, there have been substantial changes, not in the syllabi, but in teaching and attainment. These changes are shown, indirectly but conclusively, by their results. Thirty years ago large numbers of pupils found this

subject utterly and completely beyond them; fifteen years ago the population had lessened much, but was still substantial; now it is relatively small; that is, there are comparatively few pupils who have no mastery of any part of this subject. To state these facts is easier than to assign their causes. The now almost universal treatment of all the congruence and parallel propositions as axiomatic in a first reading of the subject is one cause, and the introduction of preliminary courses involving thought in addition to mere drawing and measurement is another; but it may well be that they have counted for little when compared with the improvement in actual teaching, which seems to be greater in geometry than in the other subjects. The truth is that the teaching of geometry has received more attention than the teaching of the other branches; and since there is no lessening of this close study and experiment, it may be inferred that teachers themselves are not yet content. A recent publication of the Mathematical Association [3] has suggested very drastic changes in the treatment of the subject, including a new treatment and use of the congruence and parallel axioms. Whatever the merits of the suggestions, the interest and discussion which they have aroused are all to the good.

Signs of the Times. It is somewhat remarkable that, though the geometrical unrest in this country has now persisted for at least thirty years, there has been no change in the maximum content of the standard syllabus and, at any rate until recently, no suggestion of any change. To obtain a proper treatment of the earlier work took a long time, it is true; indeed, there may still be room for improvement in this respect. But broad and large, the effects of this change can now be seen; and though they are all to the good, the teachers show, by their actions and discussions, that they think there is more to be done. Complaint is often heard, for example, that too few boys can attempt easy riders [4] with fair hope of success. Signs are not wanting that the next general movement may be in the direction of an enlarged syllabus or range of study, on the ground that the present content is too small to give an adequate background of geometrical experience. The teaching of descriptive geometry is certainly spreading, if slowly; and the study of one or two simple curves other than the circle is not now regarded as utterly impossible, as it would have been fifteen years ago. In

[3] *The Teaching of Geometry in Schools,* a Report of the British Association, 1926.

[4] In the United States we would say "originals."—Editor.

short, teachers may be deciding unconsciously that further study of the axioms, is, for the present, the flogging of a dead horse; and that the next line of experiment must be in a wider study of the subject.

Numerical Trigonometry. The work in numerical trigonometry has already been sufficiently described. It must be added, however, that this subject is not regarded as a luxury for the abler pupils only; on the contrary, it is studied by all or nearly all, and has in practice been found to be a solvent whereby many boys and girls have found a first interest in, and power over, their mathematical work. Had the subject been treated on the traditional abstract line, with the usual crop of identities and equations, the experiment would undoubtedly have failed, for it would not have provided that link between number and space which appeals to the pupils in question.

Reasons for Progress. In the ordinary work of the secondary school, then, the period under review has been one of quiet study and substantial progress as regards methods of teaching, and has been signalized by three outstanding features: the spread of logarithms, the spread of trigonometry, and the tacit refusal of teachers to limit the syllabus in algebra. It might easily be thought, and no doubt sometimes is thought, that these three features were due to ordinances imposed from above, as, for example, by Examining Bodies. Nothing could be further from the truth. As a matter of sheer history the fact is that logarithms and trigonometry found their way into many schools before they appeared in examination syllabi, and only appeared in these syllabi at all because schools told the Examining Bodies that they were doing the work and wished it to be represented in the Examinations; and had these schools desired a reduction of the algebra syllabus there is no reason to think that they would have been less enterprising in this respect. It may fairly be concluded, therefore, that during this period the teachers of mathematics have taken the first steps towards becoming a corporate body with a mind of its own, a change of even more importance than the particular events which signalize it, and largely due, no doubt, to the efforts of the Mathematical Association and its branches.

The Calculus. As to the future, the continued unrest as regards geometry has already been mentioned, but the phenomenon of most immediate significance concerns the calculus. This subject

now stands where simple trigonometry stood fifteen years ago. It is "in the air" for quite ordinary boys, and a small but increasing number of schools are actually attempting it with success, quite independently of any examination requirements, the work including differentiation and integration for rational algebraic expressions (or polynomials) at least. That history will repeat itself can hardly be doubted: the end must surely be—years hence, perhaps—that in some simple form these subjects will find a place in the First Examinations for all ordinary pupils. In a very limited sense this has indeed already happened. In most of these Examinations there is a subject (taken by very few candidates) called "Additional Mathematics," which by tradition consisted of further algebra, abstract trigonometry, and perhaps a little analytical geometry; by the act of the schools themselves, simple calculus now appears in substitution for, or as an alternative to, the bulk of this syllabus.

It is in calculus, if at all, that the formal algebra now taught will find its proper aim, as the few schools which now attempt the subject with ordinary boys have already found. In fact, it already seems likely that calculus will fuse with algebra in secondary schools, as it has in modern analysis; at least, that is what experience in this country (small, of course, as yet) suggests as the outcome.

IV. Development of Advanced Work

Second Examination. Returning now to the secondary schools themselves, one more development must be mentioned. In 1917, the Board of Education initiated a systematic encouragement of advanced work for more able pupils who remained at school for at least two years after the First Examination or Matriculation stage. Many schools had for a long time attempted such work, the pupils usually proceeding to a university; but for reasons usually financial in nature, many others could not do so, though equally willing and able for it. The encouragement has been effective: in mathematics as in most other subjects there is now a largely increased amount of this higher work. Normally two or three main subjects are taken, mathematics being continued most often, of course, with physics or chemistry or both; and a corresponding Examination called the "Second Examination" has been set up.

In mathematics, the conditions as regards the syllabus are still somewhat confused, as is perhaps only natural considering the short

lapse of time since higher work on this larger scale began. Schools had to begin by doing what they could, and often therefore took pure mathematics but no mechanics, a separation probably due to the fact that many English universities regard pure and applied mathematics as two subjects and allow the former to be taken without the latter. Where the Second Examination conditions allow it—there are eight independent Examining Bodies, each having its own regulations—this condition has tended to persist, despite a possibility of taking in combination, as one subject, lesser amounts of pure and of applied mathematics respectively. Such a combination would be easier if mechanics were a part of the normal minimum course in mathematics, but that is as yet rare; and so far as can now be judged the calculus is likely to precede it. The syllabi for higher work are therefore likely to develop independently as teaching at this level matures.

To describe the syllabi of this higher work is perhaps unnecessary. They represent, more or less, what able pupils can achieve in two years when reading mathematics as one of two or three principal subjects to which they devote the bulk of their time and include, of course, substantial amounts of all the main branches of mathematics. For this Report, the one fact of leading importance is that there is now a large number of pupils reading in this way and undoubtedly with profit; it is yet too soon to see the full result.

V. Last Development of Importance

More Intensive Work for Gifted Students. One more development, small in size but important in nature, remains to be noticed. Pupils of outstanding mathematical ability—comparatively rare, of course—are not well suited by a course in which mathematics is treated as one of two or three subjects; they need and deserve a more intensive training. This has come to be recognized, and provision has been made to meet the need. There is no fear, therefore, that this outstanding ability will be overlooked in the organization of higher education which has just been described; and the existence of the safeguard is one more evidence of activity of mathematics teachers.

VI. Summary

Progress Has Been General. It remains to be said that the limitation of this report, so far as concerns secondary education,

to the conditions in the state-aided schools must not be taken as suggesting that in other secondary schools, which include of course the public schools, there has been no corresponding progress. The exact opposite is the case; but the progress has been alike in the two cases and has been more easily described in the way chosen, since, as was said at the outset, it has been so largely conditioned and even prompted by the public organization of education as a whole.

Central Schools. Until recent years, at the school stage little mathematics has been taught in this country outside the secondary schools, the term secondary connoting any school which normally keeps its pupils until they are 16 years of age at least. There have been, however, a number of boys and girls in the public elementary schools—which are state-supported and have no fee—who were unable or unwilling to proceed to a secondary school but had, by law, to remain at school until 14 years of age at least. More or less formally according to their number, many of them have received instruction in subjects such as French, mathematics, and science; in some of the larger areas they have been gathered into "Central Schools" for a course of three or four years ending at 15 years of age.

There has thus emerged a new problem, that of providing from 11 years of age for boys and girls who are not lacking in knowledge and capacity but who do not proceed to a secondary school, perhaps because they fail to gain admission, perhaps because their parents are unwilling that they should stay at school until over 16 years of age to receive an education thought not to lead directly to good employment. It must not be thought that these children are necessarily less able than those who enter secondary schools; no doubt many of them are, but on the other hand, in large towns the parents of really able children not infrequently make this choice.

Modern Schools. It is proposed to make provision for these cases by a new type of school for which the term "Modern School" has been suggested. Those in charge of these schools will be faced with the need of evolving a new curriculum, and here, for teachers of mathematics, is a new and pressing problem. What can and should be taught to children in a course of three or four years ending at 15 years of age and followed by employment?

Not unnaturally, many or most of those who have already faced this problem, for example in the Central Schools, have resorted to

the obvious expedient of proceeding as far as possible on the lines customary in secondary schools. But that must lead to intellectual failure, for the pupils and staff are handling a subject which is truncated, not complete.

Effect of Advanced Work in Mathematics. The word "complete" has here been used not vaguely, but in a very definite sense. The growth of advanced work in secondary schools has already been described, but mention of one of the most important effects of this growth has deliberately been postponed. The reaction on the work and teaching throughout the schools, not merely in the higher classes, has been very great. In mathematics at least, the teacher cannot be thoroughly effective unless he knows a great deal more of the subject than is contained in the syllabus under which he is working; for example, an understanding of functionality and the ideas of the calculus are essential to a complete treatment of graphs, however elementary the stage. In most secondary schools, the mathematical work is now controlled by teachers who are themselves, with perhaps some of their colleagues, handling day by day advanced work of various kinds. The elementary teaching has thus been more fully informed, and to this is due much of the improvement in teaching which has already been described mathematically at least. A secondary school not having this advanced work can no longer be regarded as complete, nor can its mathematics be fully productive.

Present Problems. A Central School or a Modern School can have no such stimulus and inspiration, for school life terminates too soon. It is for mathematicians themselves to find a proper alternative, which must almost inevitably lie in a close linkage of the leading ideas and process of mathematics with matters arising in science, industry, commerce, and social life. There seems to be no reason why boys and girls at this age should not acquire the power and habit of thinking mathematically on any occasion that may be appropriate, and some teachers are, indeed, conscious that this is the main need in Modern Schools and they are attempting to meet it. The subject (or method) known as practical mathematics, which thirty years ago did so much for teaching in this country, is in mind in this connection; but for children of the age in question systematization and the introduction of a larger amount of specific training would be regarded as necessary by many or most of those who are concerned at first hand with the actual teaching.

Obviously, the problem is not easy of solution, if only because a fairly complete break with tradition is necessary. The present generation of teachers may, indeed, fail to find even the beginning of a proper solution, owing to the handicap of their traditions; but they are certainly conscious of the need, and the discussions and trials of the next few years are likely to make an interesting contribution to the history of mathematical education.

FRANCE

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Organization of Education in France. As is well known, education in France is organized almost entirely on the lines of the official program of studies: on the one hand, public or state education absorbs the large majority of children and young people; on the other, the independent educational organizations are obliged to adapt themselves to the official curriculum in spirit, if not to the official time-tables, in order to prepare their pupils for examinations and competitions which are the same for all candidates of whatever origin and which are passed before state examination boards.

Therefore an indication of the changes in the curriculum, with the addition, if necessary, of certain tendencies expressed in periodicals or recent books, should, in my opinion, suffice to constitute a short survey of the evolution in the teaching of mathematics in France during the last eighteen years. I shall begin by recalling briefly the main branches of education in our country.

Infant Schools. Instruction for children of from 2 to 6 years is given in infant classes, or in "kindergartens," either public and free, or annexed to the secondary schools for girls (and, but more rarely, for boys), or private and paying.

Elementary Education. For children from 6 to 11 years of age, elementary education is divided into a preparatory course (one year), an elementary course (two years), and an intermediate course (two years). It is compulsory, and is free in the "public schools," of which there is at least one in each commune. For those children not intending to continue their studies, these courses are completed by two years of higher courses preparing for the Certificate of Elementary Studies (*Certificat d'Etudes Primaires*) the examination for which takes place in every district (canton).

Intermediate Education. For young boys and girls intermediate education comprises three main branches:

1. Technical education, from 11 to 15 years of age, comprising one year of higher studies and three years of instruction in manual and intellectual subjects. This is given in Schools of Practical Commerce and Industry (*Ecoles pratiques de Commerce et d'Industrie*), and also in a certain number of Trade Schools (*Ecoles de Métiers*) and National Technical Colleges (*Ecoles nationales Professionnelles*). Such schools prepare for a certificate in practical commercial or industrial studies.

2. Higher elementary education (*Enseignement Primaire Supérieur*) also comprises one year of higher studies and three years of general studies to which are added, whenever possible, a few hours of manual work per week. This instruction is given in the Higher Primary Schools, or in the *Cours complémentaires*, and prepares for a Certificate of Higher Elementary Studies (*Brevet d'Enseignement Primaire Supérieur*), with several optional subjects: agriculture, industry, housekeeping, and the like, the "*Brevet Simple*"; the examinations take place in each county (department).

3. Secondary school course of seven years (from the sixth to the first form, plus one class in philosophy or mathematics) in *collèges* and *lycées*, preparing for the *Baccalauréat*, an examination which is passed in two successive years before entering the University.

Other Phases of Intermediate Education. Courses also exist for young boys and girls who are occupied during the day and who cannot attend the schools mentioned above. These are "Adult Courses" and "Trade Classes." The Adult Courses have no determined curricula and provide a continuation of the general instruction in the elementary school. The Trade Classes, which the authorities are trying to make compulsory, have much the same object in view as that of the Adult Courses, but they aim more distinctly at practical and technical preparation for a given profession.

Courses for Completing Intermediate Education. Certain courses of instruction with a view to completing the intermediate education are offered as follows:

1. Technical Instruction is continued in the Schools of Arts and Crafts (*Ecoles d'Arts et Métiers*, seven national schools and similar independent establishments). Admission is by a difficult competitive examination; the instruction is partly theoretical, partly manual; and after three years' attendance, the students graduate as engineers.

2. Higher Elementary Instruction is continued in the Training Colleges (*Ecoles Normales Primaires*, one to each county); admis-

sion is by competition and the object is to train teachers in three years of general studies and practical pedagogy.

3. Higher Elementary Instruction and Technical Instruction are continued in various special schools of commerce, electricity, etc., both public and private.

4. Secondary Education is completed by two courses of preparation for the *Grandes Ecoles* or by pre-university studies: the *Première Supérieure* for arts and *Mathématiques Spéciales* for sciences. Similar instruction is also given in the Faculties of Sciences in preparation for the certificates of *P.C.N.* (physics, chemistry, natural history), *M.P.C.* (mathematics, physics, and chemistry), and general mathematics.

Higher Education. (University Instruction.) This is given principally in the Faculties of Arts and Science preparing for the various licenses and doctorates, and also for the competitions for secondary school teachers (*certificats d'aptitude, agrégations*). It exists also in certain *Grandes Ecoles,* such as the *Ecole Normale Supérieure, Ecole Polytechnique, Ecole des Mines, Ecole des Ponts, Ecole Centrale,* and so on.

The Place of Mathematics in Each Division. In this rapid summary I have made no distinction between boys and girls; the curricula and competitive examinations, although still differing on some points, tend to become identical. In going over these main divisions, my object will be to show the place given to mathematics and to explain their development.

Infant Schools. There is, strictly speaking, no official curriculum for these schools, but numerous tendencies are noticeable in the curriculum of the certificate for teachers in infant classes (attached to Girls' Secondary Schools) in the teachers' congresses, and also in pedagogical magazines.

The first teaching of, or more exactly, the first instruction in mathematics, comprises, above all, a concrete conception of numbers obtained by comparing different groups of objects. The child acquires a notion of addition and multiplication also by experimental study of the various partitions (or divisions into sums) of simple numbers. Certain definitions, forms, and even geometric principles are learned by sense training and suitable apparatus. I wish to note particularly M. Terquem's very ingenious apparatus; it consists of pieces of cardboard of several colors, of very simple figures, equilateral triangles, rhombuses, regular hexagons, squares, and half-

squares; their outside dimensions arranged in the ratio $\sqrt{2}$ permit of their being put together or superimposed in an almost unlimited number of ways.

Elementary Schools. A revision of the curriculum made in 1923 (Decision of February 23, Instructions of June 20) modified the mathematical section only slightly, and is marked chiefly by a few simplifications and a tendency to more concrete teaching. The adoption of the metric system, which is now general in France, has made it possible to do away with the study of general fractions and limit instructions entirely to decimal numbers, deduced from the consideration of multiples and submultiples of legal measures. Geometry remains, as heretofore, a study parallel to drawing, which is nevertheless supplemented by the numerical application in rules for surface and simple volumes. But it is not unlikely that the use of the Terquem apparatus, or something similar, may create a closer link between the teaching of geometry in infant classes and elementary schools, thus giving to intuitive and visual geometry a larger and more important place in the methodical development of observation, reasoning, and intelligence in children.

Lastly, in the higher classes (11 to 13 years) the study of fractions is continued, and the use of letters for the solution of problems of the first degree has been somewhat tentatively introduced. This leads, of course, to the immediate stating of the conditions of the problem in the form of an equation or a system of equations and the solving of it by successive simplifications of the equation or system thus obtained. On the contrary, the so-called arithmetic method must substitute for the enunciation an equivalent enunciation (more or less easily obtained without the help of calculations) which leads to an equation or a system which can be solved immediately and which consequently need not be written out in full.

It must be borne in mind that this curriculum, having been in force for five years only, has not been universally applied as yet, either in letter or in spirit, and certain of the modifications indicated can be put into effect only slowly. A first inquiry to be made throughout France regarding the application of these modifications was planned for the pedagogical conferences of elementary school teachers, which meets in October and November in each district (canton). The questions they were to attempt to answer comprise:

1. The use of the addition table, memorized, or based on simple remarks on the partition of numbers.

2. The possibility of a brief theoretical teaching of the rules of multiplication and division.

3 and 4. The study of decimal fractions along with that of ordinary fractions based on simultaneous or previous knowledge of the metric system.

5. The solution of problems (typical problems, analytic methods, trials, use of letters, etc.).

6. Curriculum for geometry in the higher forms.

Technical Education. The programs have not been modified since 1909 and may be compared, in letter and spirit, to those of the higher elementary education of which I shall speak later. Nevertheless, there is a growing tendency, in all quarters, towards technical instruction with a view to a trade and by means of that trade (*La pédagogie de l'Enseignement Technique*, L. Eyrolles, ed., 1927). Besides these programs there are the Instructions for 1909: "The object of algebraic calculus is to enable the student to understand and apply the formulas he will meet with in courses in mechanics, technology, and electricity, and, later, in the formularies." In so far as I can judge from the books in use, and certain classes and results that I have seen, there is still room for improvement in the closer linking up of theory and current practice, in order not to separate descriptive geometry from the drawings of machines, geometry from the workshop problems, algebra from commercial questions, mechanics from physics, or even from everyday experience. But this effort is not beyond the capacity of the great majority of teachers.

Higher Elementary Education. This branch was originally merely a continuation of elementary education and therefore comprised only arithmetic, or what was called theoretical arithmetic, divisibility, prime numbers, and the theory of the working of operations. The exercises consisted of problems of the first algebraic degree, to be solved arithmetically; that is, without the use of letters, as I have indicated above for the higher courses. As early as 1909 (*Décret* and *Arrêté* of July 26) the curriculum comprised algebra (systems of equations of the first degree, numeral equations of the second degree) and the study of plane and solid geometry limited to the essential definitions, to angular and metric relations, and to a few demonstrations of the formulas for areas and volumes. But arithmetic maintained, of course, part of its former importance and the instructions indicated that: "problems in arith-

metic, the educative influence of which is not unimportant, shall occupy the chief place in the Higher Elementary Schools." This first indication was, as a matter of fact, corrected by a second one (which has not been unfailingly observed) that: "questions of a purely speculative order must be done away with, and that there must be substituted for certain 'absurd' problems on mixtures and alligation, exercises on the rations necessary for the feeding of cattle," and so on. The exercise of mental arithmetic was also devised as "excellent training towards suppleness and quickness of mind in dealing with mathematical questions." (I highly approve of mental arithmetic, but for other reasons.)

Release from Arithmetic Tradition. In 1920 another effort was made to release the curriculum from arithmetic tradition. It was clearly specified, as in elementary education, that the study of decimal numbers, based upon the use of the metric system, should precede that of general fractions. A handbook, the author of which, M. Millet, formerly a teacher in the higher elementary schools is now *agrégé* teacher in a *lycée*, furnishes a characteristic example of this method (Hachette, 1923). The method of abbreviated and symbolical notation is advised in the working out of questions, and the following is added to the Instructions of 1920, which are still in force: "From the beginning the student shall be encouraged to use the letter notation and be initiated to simple numerical algebra; the utility of this will have its influence on the study of questions which up until now have been worked out by arithmetic." (I may be permitted to express some doubt as to this subtle distinction between reasoning by algebra and by arithmetic.) Above all, the use of graphs is advised, and in the third year there is introduced tentatively the study of functions x^2 and $\frac{1}{x}$. It is possible that such studies, carried out in a sufficiently concrete and intuitive way, might precede, or at least elucidate the study of square root, equations of the second degree, and inversely proportional quantities.

Geometry Curriculum. The geometry curriculum has remained practically unchanged; it should not be separated from observation and practice, or from drawing and manual work. Unfortunately, no indications are given regarding exercises; the necessity of preparing their pupils for examinations has kept the teachers to the "speculative" problems, and has perhaps prevented the strict application of the Instructions.

Secondary Education. The programs for 1900, even with the modifications for 1905 and 1909, had drawn a marked distinction between the "literature classes" (6th to 3rd A; 1st and 2nd A and B) and the "science classes" (6th to 3rd B; 1st and 2nd C and D). The program for 1925 is characterized, among other things, by a program of mathematics and sciences common to all candidates for the first part of the *baccalauréat*. The official instructions take into account the difficulties of this organization: "Students of frequently varying standards will be for a period of six years under the same instruction. For the teaching to be as fruitful as is hoped, it is indispensable that the classes remain as homogeneous as possible. Such a condition is only possible if the great majority of students are interested: therefore, the teaching must be understandable to that majority. Simplicity and clearness are necessary." This *amalgam* has found ardent opponents as well as ardent partisans; it was apparently proposed by those whose competence rested on no scientific foundation. However that may be, the results of the experience will not clearly appear for several years to come, and it will not be until July, 1929, that the candidates to the various *baccalauréats* (first part) will come up for a common scientific test.

As to pedagogical details, the Instructions emphasize the necessity for work in common in the classroom, of an association between pupils and teacher for the working out (or, to use an accepted term, the "rediscovery") of problems; they also emphasize the necessity for obtaining an "understanding" of mathematics, and of shortening as much as possible the period of blind submission to imposed rules, thus facilitating an "awakening of the critical sense."

This would seem to be in opposition to the doctrine of technical and higher elementary education. The opposition is perhaps greater in theory than in reality. The teachers of various grades are now passing in increasing numbers through the universities; the *licence mathématique* [1] (certificate of differential and integral calculus, rational mechanics, general physics) is the degree which is most generally sought. While striving to adapt themselves, all the more

[1] Teachers in French secondary schools (*lycées* and *collèges*) are either *agrégés* who are selected scholars who have passed a very difficult competitive examination known as the *agrégation*, or *licenciés* who have obtained the *licence* which corresponds to the degree of Master of Arts. The examination for the *agrégation* is more severe than that for the degree of Doctor of Philosophy in the United States.—Editor.

successfully for their being more learned, to the special objects their pupils have in view, the teachers maintain and communicate to their classes the clear logical spirit, the desire for proof, and even that slight skepticism which is typical of French mathematicians. In all grades of teaching, demonstration is the rule; hard and fast truths, unexplained formulas are nearly always forbidden; exceptions are studied with as much interest as general cases.

The Class of Mathematics, which is parallel to the Class of Philosophy, and brings secondary studies to an end, is still on the lines of the old curriculum, although trigonometry and descriptive geometry have been added, having been eliminated from the preceding class; dynamics alone has been done away with.

Supplementary Education. There is a tendency to raise the level of mathematical studies in the entrance examinations and in the curricula of the schools of arts and crafts. In the training colleges (*Ecoles Normales Primaires*), the *Arrêté* of 1920 demands above all the development of mathematical knowledge acquired in the higher elementary schools. Its interpretation varies greatly in different districts, however, especially for the third year, in which surveying, cosmography, and descriptive geometry are not officially recognized.

In the Class of Special Mathematics and in the certificates of general mathematics of the various universities, analysis (derivatives, integrals, series, differential equations, equations in real variables) has been developed at the expense of pure algebra (theory of equations) and especially of analytic geometry and modern geometry. Vectorial calculus has been introduced tentatively into the official curriculum, but certain recent books and a general tendency of thought will doubtless bring about their more common use.

University Education. Candidates for the *Licence* are always at liberty to prepare for the separate *Certificats d'Etudes Supérieures* of differential and integral calculus, the curriculum including the theory of analytic functions and differential equations; of rational mechanics, kinetics and dynamics of solids and systems of solids; and of general physics. To these the students may add, if they wish, special certificates in advanced analysis, advanced geometry, applied mechanics, astronomy, mathematical astronomy (*mécanique céleste*), and so on, the curriculum and teaching of which vary with the different professors and lecturers.

I can see no important modification here, any more than in the

doctorate, which remains a degree of a high order given for personal work of value.

I have already pointed out that the preparation of teachers of mathematics in practical schools, higher elementary schools, and in *collèges* (secondary schools in small towns) tends to become uniform through the *Licence,* which is, be it said, more strictly regulated than in the past, specified certificates being required. But there are still special teachers' certificates for teaching in the practical schools on the one hand, and in the higher elementary schools on the other. Lastly, the supply of *lycée* teachers (though their teaching is the same as that of teachers in *collèges*) is assured amongst *licenciés* by the difficult competitive examination of the *agrégation,* where the percentage of "passes" is only 25 per cent. In this connection the increasingly higher level of the *agrégation* for women is noteworthy; this now consists of three papers—in elementary mathematics, in algebra and analysis, and in geometry and mechanics. Nevertheless, it is hardly possible as yet to foresee the complete assimilation of this examination with that for men.

GERMANY

By DR. W. LIETZMANN

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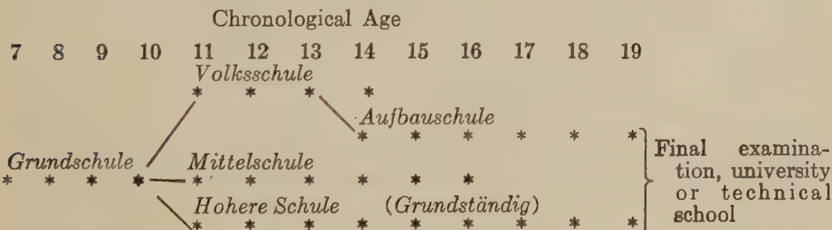
I. General School Organization

Important Changes. A knowledge of the changes in the structure of the German school system since the time of the IMUK[1] report is a necessary prerequisite to an understanding of the present condition of the teaching of mathematics. Discussions regarding the school curriculum which had begun before the close of the World War and which led to stormy controversies during the Revolution, reached a more peaceful stage after a *Reichsschulkonferenz* in 1920. At this educational parliament, with its more than six hundred members, certain fundamental principles were agreed upon. The discussions centered about the questions of the training of teachers, the organization of the schools, and the method of instruction called the *Arbeitsunterrichts*.[2] The national government took the initiative in legislation by founding a four-year *Grundschule* which was to be compulsory for all children, but the further regulation of education was soon surrendered to the states. We shall have time to describe only the situation in the greatest of the states, i.e., in Prussia, and to note a few variations from it.

General Structure. The following diagram indicates the general structure of the Prussian school system:



[1] The *Internationale Mathematische Unterrichts-Kommission*, known as the IMUK, is the German name for the *International Commission on the Teaching of Mathematics* whose work terminated in 1922 with the publication of its last report under Professor Fehr of Geneva.—Editor.

[2] The *Arbeitsunterrichts* is explained and discussed on page 45.—Editor.

The Grundschule. All children are required to attend the *Grundschule,* which has a four-year course. Following this, the pupils may enter a four-year *Volksschule,* which is intimately connected with the *Grundschule* in its administration, or they may attend a six-year *Mittelschule,* or lastly, they may go to a *grundständige Hohere Schule,* which has a nine-year course. The *Mittelschule* prepares for all sorts of vocations; it requires one foreign language, offers nearly as much mathematics as the first six classes of the *Hohere Schule,* and sometimes allows the better pupils the opportunity of transfer to them. The higher schools prepare for the examination which is required for entrance to the universities, the technical schools and the other advanced institutions. Counting from the beginning of the *Grundschule,* this course is thirteen years long.

The Aufbauschule. Gifted students in the *Volksschule* may be able to reach this same goal in a six-year course in the so-called *Aufbauschule* taken after their seventh school year.

The Mittelschule. The *Mittelschule* is found only in Prussia and in a few cities in north Germany, and *Aufbauschulen* have not yet been established in certain of the south German states, as for example, in Bavaria.

Types Before the War. Before the War, there were three types of secondary schools in Germany: (1) *Gymnasien,* with Latin, Greek, and one modern foreign language. (2) *Realgymnasien,* with Latin and two modern foreign languages. (3) *Oberrealschulen,* with two modern foreign languages.

Variants of these were found in the *Reformgymnasium* and *Reformrealgymnasium,* in which that purpose was the same as that of the *Gymnasium* and the *Realgymnasium* but in which Latin was replaced by a modern foreign language as the first foreign tongue to be studied. Shortly after the War, a fourth type of school was organized in Prussia and in a few other states. This was the German *Oberschule.* It resembled the *Oberrealschule,* but differed from it in subordinating the second foreign language to German and history rather than to mathematics and science. The six-year *Aufbauschule* corresponds to the German *Oberschule* but not to the *Gymnasium* and the *Realgymnasium.* Coeducation does not exist in Germany. With some slight variants, secondary schools for girls follow the same form types as those for boys, but girls' schools corresponding to the *Gymnasien* are rare.

Training of Secondary Teachers. The training of teachers for the secondary schools continues as before; an examination (*Reifeprüfung*), at least four years' study at a university (for the mathematician this may be at a technical school), a state examination, one or two years of practical, general pedagogy and methods of teaching in a normal school (*Seminar*) usually attached to a secondary school, and lastly an examination in pedagogy. The preparation is adjusted according to supply and demand. In former times, teachers of the *Volksschule* and *Mittelschule* completed the *Volksschule* and then attended a normal school for six years. These normal schools have now been given up in Prussia and in a few other states, but they are still to be found in Bavaria. Prospective teachers of the *Volksschule* are now required to pass the *Reifeprüfung* and to take two years of training in a *Pädagogischen Akademie,* of which Prussia has four; or in *Instituten,* as in Saxony; or in the university itself, as in Thuringia and in Hesse.

II. The Curriculum in Mathematics

Unified Sketch Impossible. Since the determination of the curricula is the business of the individual states, it is impossible to give any unified sketch of the work in mathematics. The variety is too great, particularly if one considers the minor differences in schools of the same type. An additional difficulty lies in the fact that instead of having a detailed plan of study which shall be obligatory in all schools, Prussia has recently adopted the scheme of issuing a standard *Richtlinien* upon which the individual institutions are to base their own plan of work. The differences in the work in mathematics between schools, however, are not so large as to prevent the discussion of certain general objectives. For example, the time devoted to the subject varies from three to six hours a week, these hours being of 45 minutes each, the total number in a week ranging from 30 to 36.

Contents. The purpose of the work in mathematics in the *Grundschule* is the verbal and written mastery of the four fundamental operations with whole numbers.

In the *Volksschule,* fractions are studied together with their applications to practical problems based on the mathematics of the home, the factory, and the state (for example, the Rule of Three and percentage). The most important concepts of both plane and solid geometry are studied and those useful in daily life and in the

simplest industries are represented clearly and an opportunity is given for measuring areas and volumes, and for drawing ground plans and elevations.

The *Mittelschule* generally offers the same work in mathematics as that given in the first six classes of the higher schools. The higher schools are divided on the 3-3-3 plan: the *Unterstufe,* the *Mittelstufe,* and the *Oberstufe.* In the first of these divisions, the work in arithmetic is planned to show the uses of the fundamental operations with whole numbers and fractions as they occur in civil and in mercantile affairs. In the *Mittelstufe,* the operations of raising to powers, finding roots, and finding logarithms are developed in connection with numerical work in the study of equations of the first and second degree, where at the present time, the ideas of functional dependence and graphic representation play leading rôles. In the last three years of the course, the number concept is extended to include complex numbers and sometimes also De Moivre's Theorem, and the study of arithmetic and geometric progressions is taken up in connection with the reckoning of interest, annuities, and sometimes also with insurance. Combinations and probability are given much less attention than was formerly the case. The idea of functional dependence is strongly emphasized and it is studied by the methods of the calculus. In the *Realschule,*[3] the differential calculus (rational integral functions and trigonometric functions) is introduced in the third year from the last, because of its value in the study of mechanics. Sometimes the integral calculus is taught at this time also. The study of equations consists entirely of the investigation of the zero value of integral functions, and solutions by Cardan's Method are being replaced to an increasing extent by methods of approximation such as the Rule of False and Newton's Method. This is followed by the discussion of rational integral, rational fractional, and algebraic functions, and the most important transcendental functions (trigonometric, cyclometric, exponential, and logarithmic). In the integral calculus, attention is concentrated on the simplest integrations, although in the *Oberrealschule* in southern Germany this work is carried considerably farther. In the *Realschule,* the simplest power series are studied as being the most useful method for the numerical mastery of algebraic and transcendental functions.

The elements of plane and solid geometry and of plane trigo-

[3] The modern school as distinguished from the classical one.—Editor.

nometry have already been studied in the *Mittelstufe*. In the *Oberstufe*, spherical trigonometry is introduced together with its simplest applications to geodesy and astronomy. Plane analytic geometry is studied also, but solid analytic geometry is seldom offered. In the *Oberrealschule* the synthetic geometry of conic sections includes the methods of Apollonius (properties of the focus, Dandelian spheres) and Desargues (methods of perspective, Pascal's Theorem). From the *Mittelstufe* on, map-making and the representation of solids both in orthogonal projection and in central perspective form a part of the instruction in geometry.

III. General Comment on Methods of Instruction

Various Plans. The plans of study in the German schools have always laid great stress on the making of "methodical notes" and the Prussian *Richtlinien* of 1925 considers these as being of prime importance. The present slogan is *Arbeitsunterricht*. This term is differently interpreted. The essence of the matter is that the pupil shall make the subject matter his own by his own independent work. Thus, the idea includes intellectual creativeness. Some would add spontaneity also. In accordance with these ideas, the teacher is forbidden to use a dogmatic exposition of his subject, and the question and answer method is being thrust into the background since it depends on strongly suggestive guidance on the part of the teacher. The more common form of instruction is an interchange of discussion between pupils and teacher. Fanatics urge discussion on the part of the pupils with the teacher almost entirely excluded. Teachers of mathematics are in violent disagreement in the arguments for and against this *Arbeitsunterricht*. They are expected to conform to it, but there are marked differences in its applications.

Problem Solving. The requirement that the pupil be intellectually independent results in a considerably increased emphasis on problem solving in mathematics. In former times, the collections of problems were limited to traditional materials such as numerical equations and problems involving equations. In geometry, they were constructions. Now emphasis is laid on examples which develop the subject matter and which teach its applications. Great progress is to be noted in the variety of examples. It is of course necessary that such a method should pay the greatest attention to the mental development of the pupils. Consequently, the student progresses from visual work and the more or less empirical handling of mate-

rial as in drawings and models to methods that are preëminently logical and deductive. In short, the material must be as real as possible in its applications; it must not use artificial settings for its problems but must take them from the usual surroundings of students of the age concerned.

Correlation. An important and constantly reiterated purpose of the *Richtlinien* is connected with concentration. The strain of a variety of studies pulling the student in many directions must be lessened by correlations between these subjects whenever possible. Not only shall mathematical problems borrow their material from physics, from geodesy, and the like, but a connection must also be made between such different subjects as mathematics and languages. In the *Gymnasium*, excerpts from Euclid and Archimedes, for example, are to be used as sources; in the *Relgymnasium* Descartes is a possibility. Instruction in the mother tongue should give attention to mathematical works, and in return classes in mathematics must pay due heed to expression in German.

Cultural Values. The *Richtlinien* lays particular emphasis on the stressing of cultural values. In the case of mathematics, this implies greater attention to historical development and also a consideration of the relation of mathematics to philosophy. Not only can logic and the general philosophic studies be utilized to this end, but also psychology and, in the investigation of the fundamentals of mathematics, the theory of knowledge. The strong emphasis laid on the history of mathematics is a tendency in the same direction. Here, the mere citation of names and dates is not sufficient, but the history of a problem is investigated and traced to its source wherever possible.

Elective Courses. The question may be raised as to how the limited hours of study can permit the attainment of such widespread aims. The general intention of this report is that a limited number of problems be considered in great detail, enough topics being omitted to make this possible. Such a procedure would be difficult in mathematics where one must lay a foundation for all the work which is to follow. Another solution has been offered, however. Several decades ago, a movement began which permitted the students in the upper classes to choose between several alternatives in their studies. This method as it is used in Lubeck makes only about two-thirds of the weekly hours compulsory for all students, the rest of the time being given to electives. The group

system is in use in Saxony. Here the pupils of the *Oberstufe* are divided into two sections: the language-history group and the mathematics-science group. Even before the appearance of the *Richtlinien*, parts of the work were elective in Prussia. The *Richtlinien* itself provided that from six to twelve hours a week be set aside for free work groups, the time varying according to the size of the institution. Choice of these work groups is made every half-year. Naturally, the students who are particularly interested in mathematics choose this subject if group work in mathematics is provided. Thus, much of the work that was mentioned above as desirable may be handled here, and in addition the work groups may treat various other topics of mathematical importance. Among these may be mentioned nomography, mathematical statistics, mathematics in relation to art, mathematical fallacies, and mathematical recreations.

IV. A Few Special Problems of the Teaching of Mathematics

Illustrative Examples. Since it is impossible to give a full discussion of the individual problems that are met in the various parts of the work in mathematics, a few examples must serve here.

Arithmetic. In the work in arithmetic in the lower classes, preparation is made as soon as is possible for the later use of literal symbols in reckoning, as, for example, in general arithmetic problems (commutative law of addition and multiplication, fractions and per cent). On the other hand, the later instruction in arithmetic carries the drill in numerical calculation into the highest class. Abridged reckoning is used on all possible occasions. An appearance of accuracy out of all proportion to the result is taboo. Four-place logarithms are used. The slide rule is used throughout, being especially essential for practical instruction in physics. Hankel's permanency of mathematical laws is constantly followed in the development of the idea of number and only in the highest classes or in the mathematical work groups is there any use made of the simplest concepts of the *Mengenlehre*. The use of Dedekind's "cuts" is also very infrequent and it is limited to the upper classes. On the other hand, more and more people are urging an earlier preparation for the idea of a series and of a limit as in square root, π, and geometric progressions.

The Function Concept. The function concept and graphic representation are used from the very beginning. The work in

arithmetic provides geometric illustrations and a few general properties of functions are worked out on the basis of empirical functions. The study of proportions is based on the work with the linear function $y = ax$. Rational integral functions, rational fractional functions, and simple algebraic functions are discussed, and among the transcendental functions, the logarithmic, cyclometric, exponential, and logarithmic functions are treated numerically and graphically.

The Calculus. In recent years, mathematicians both of the universities and of the secondary schools have taken part in discussions in regard to the teaching of the calculus. A considerable accuracy in introductory analysis is gained if one begins with the consideration of functions, treating them from the point of view of geometry and of physics. Informal discussions are utilized where the rigorous treatment is beyond the capacity of the upper classes, as for instance in the differentiation of a power series and in the development of the *Restglieddiskussion.* This expedient is not a new one for it was formerly the regular custom to use the fundamental principles of algebra in high school classes without explanation.

Complex Variable. A novel feature of the Prussian *Richtlinien* lies in the suggested use of the functions of a complex variable. Where this suggestion is followed, it is sufficient to consider the linear integral and the linear fractional functions of a complex variable (incidentally introducing the concept of Riemann's surfaces). This opportunity is utilized to give an outlook into the realm of the simplest transformations. Other teachers give the concept of transformations by means of Klein's Erlanger program, omitting the complex variable.

Propædeutic Methods. The propædeutic introduction to geometry has already proved its value and accordingly it has been retained. Practical mensuration is utilized to lay a foundation for the determination of capacities and volumes. In the guise of field surveying, it accompanies the study of areas and similarity. And in the measurement of heights and surfaces, it is used in trigonometry and in analytic geometry. Spherical trigonometry is taught everywhere, although in the *Gymnasium* it is limited to the simplest theorems needed for the mathematical study of the earth and the heavens,—a study which the student may make with a theodolite.

Measurements. The measurements of plane surfaces and of volumes are combined under the heading of proportions in some-

what the same way as the computation of volumes has previously been combined with the study of similarity.

Fusion. The "fusion" of subject matter is also aided by the graphic representation of solids. Even in the *Mittelstufe,* the Prussian *Richtlinien* suggests that simple solids be studied according to the *Eintafel* method discovered by Scheffer, a teacher of mathematics in *Technische Hochschule.* By his method, the numerical values of dimensions are represented by lines placed side by side. The course continues to orthogonal and oblique projection and to the making of ground plans and elevations. So far as possible this is accompanied by central perspective and map-making. In the applications of spherical trigonometry to mathematical astronomy, it is now customary to use the methods of constructive geometry in parallel with the methods of pure computation.

Use of Geometric Methods. The use of geometric methods has a valuable application to the study of conic sections in the *Realschule* at least. The connection between planimetric and stereometric definitions of conics given in the concepts of Apollonius and the union of the two concepts in work with the Dandelian spheres, permits the old Euclidean methods of deductive reasoning to prove their value in the study of the properties of the focus. The contributions of Desargues are of use here also and a connection is made with descriptive geometry as well. In exceptional cases, analytic geometry is used to give a third and novel method of treatment. The study of conics through the method of projections has practically disappeared from the secondary schools, although it formerly had the support of a few enthusiasts among German teachers of mathematics.

Scope. The outsider may perhaps be surprised at the scope of the curriculum in mathematics. But he should realize that this course should not be compared with that of the ordinary high school but that it practically includes the first two years of the traditional college course as well. Furthermore, not all of this subject matter is compulsory. In one place one topic is omitted; in another, something different. In schools of the *Gymnasial* type more is omitted: in schools of the *Real* type, less. In addition to this, the tendency to bring the individual parts of mathematics out of their former isolation and to force them into a single whole gives each part the support of the rest—a thing that does not follow when these topics are treated either in parallel or in successive courses.

Result of Klein's Influence. Let me add one point in conclusion: The present condition of the teaching of mathematics in Germany is a direct outcome of the reform begun under the leadership of Felix Klein in 1905 under the name of the *Meraner Vorschlagen.* The German subcommittee of the IMUK provided the necessary impetus. The revision of this curriculum, begun in 1917 at the request of the Prussian Minister of Education and published in 1922, summarized the reforms proposed in the intervening years. The Prussian *Richtlinien* and the syllabi of the other German states have been based on the recommendations of this plan and with a few additions they have been made generally compulsory.

V. Bibliography

Selected References. I am obliged to confine this bibliography to but few works. The periodicals devoted to the mathematics of the secondary schools are:

Zeitschrift für den mathematischen und naturwissenschaftlichen Unterricht, printed by B. G. Teubner, Leipzig, and edited by H. Schotten, W. Lietzmann, and W. Hillers.

Unterrichtsblätter für Mathematik und Naturwissenschaften, printed by O. Salle, Berlin, and edited by G. Wolff.

The literature dealing with the organization of the curriculum in mathematics, the course of study, and the methods of instruction includes:

W. Lietzmann, *Methodik des mathematischen Unterrichts,* Vol. I, second edition 1926. Quelle and Meyer, Leipzig.

The following articles also give a brief survey:

W. Lietzmann, "New Types of Schools in Germany and Their Curricula in Mathematics," *Mathematics Teacher,* XVII (1924), pp. 148 ff.

F. Malsch, "The Teaching of Mathematics in Germany Since the War," *Mathematics Teacher,* XX (1927), pp. 355 ff.

Mathematical subject matter and its method of presentation will be found in:

W. Lietzmann, *Methodik des mathematischen Unterrichts,* Vol. II, 1923, and Vol. III, 1924. Quelle and Meyer, Leipzig.

From the long array of works on the modern curriculum in mathematics, many of which are in several volumes and which

often are in several editions for the different types of schools, but which deal with the entire course of the secondary schools, I will name:

Lietzmann	Teubner	Leipzig
Götting-Behrendsen-Hamauk	Teubner	Leipzig
Schülke-Dreetz	Teubner	Leipzig
Heinrich Müller	Teubner	Leipzig
Mannheimer-Zeisberg	Diesterweg	Frankfort am Main
Zacharies-Ebner	Diesterweg	Frankfort am Main
Malsch	Quelle and Meyer	Leipzig
Reidt-Wolff-Kerst	Grote	Berlin

In conclusion, I shall cite a few publications dealing with particular problems of instruction. Separate topics, some giving the historical point of view and others emphasizing content, are discussed in the little volumes of the *Mathematisch-physikalischen Bibliothek*. These books, about seventy in number, are very popular in the schools. They are edited by W. Lietzmann and A. Witting and are published by Teubner (Leipzig). Another series is the new *Mathematisch-naturwissenschaftlich-technischen Bücherei.* Sixteen volumes of this have appeared to date. The editors are G. Wolff and E. Wasserloos. The publisher is Salle in Berlin.

For the enrichment of mathematical instruction offered by philosophy, I list:

W. Lietzmann, *Erkenntnislehre in mathematischen Unterricht der Oberklassen.* Mundus-Verlag, Charlottenburg, 1921.

W. Lietzmann, *Aufbau und Grundlage der Mathematik.* Teubner, Leipzig, 1927.

The many volumes of the two collections mentioned above are useful for their historical content and recently published textbooks include historical notes and introduce problems of varied types taken from the original sources. In this connection, see also:

W. Lietzmann, *Ueberlich über die Geschichte der Elementarmathematik,* 2nd edition. Teubner, Leipzig, 1928.

For the teacher's use, the best help is the second edition of the seven volume *Geschichte der Elementarmathematik* by J. Tropfke (Vereinigung wissenschaftlicher Verleger, Berlin, 1921).

I shall name but two special monographs. The most useful work for the study of *Eintafel* projection is:

G. SCHEFFERS AND W. KRAMER, *Leitfaden der darstellenden und räumlichen Geometrie,* Vol. I, 1924; Vol. II, 1925. Quelle and Meyer, Leipzig. See also the volumes by Balser and Kramer in the *Mathematisch-physikalischen Bibliothek.*

For technical applications, see:

M. HAUPTMANN, *Technische Aufgaben zur Mathematik.* Teubner, Leipzig.

See also Rothe's volume in the *Mathematisch-physikalischen Bibliothek.*

HOLLAND

BY DR. D. J. E. SCHREK
Utrecht

Introduction. There are reasons that make a survey of the changes and trends in the teaching of mathematics since 1910 desirable. The report, issued in 1911 by the Dutch Branch of the International Commission,[1] is now quite out of date, and so are more or less of the communications about Holland in various later American reports.[2] The latest article on this subject is, as far as I know, a contribution by the author of this paper to the *Mathematics Teacher*,[3] which the reader may consult for further information. A list of current Dutch mathematical textbooks will be found in the first part of Dr. Lietzmann's well-known work on the teaching of mathematics.[4]

I. EDUCATION IN GENERAL IN HOLLAND

Dutch Organization of Education. First of all it is necessary to make some introductory remarks on the Dutch organization of education in general. Elementary instruction is given to children of about 6 to 12 years of age, the schools including six or seven forms. These are the so-called *Lagere Scholen* (lower schools), the work being compulsory. Sometimes an elementary school is connected with three or four additional forms, supplying *Meer uitgebreid lager onderwijs* (M.U.L.O.), i.e., more advanced instruction, including mathematics and foreign languages, but in such a case

[1] *Rapport sur l'Enseignement mathématique dans les Pays-Bas, publié par la Sous-Commission nationale de la Commission Internationale de l'Enseignement Mathématique.* Delft, Waltman, 1911 (151 pp.)

[2] E. g., J. C. Brown. *Curricula in Mathematics* (U. S. Bureau of Education, Bulletin, 1914, No. 45) and R. C. Archibald. *The Training of Teachers of Mathematics* (U. S. Bureau of Education, Bulletin, 1917, No. 27).

[3] D. J. E. Schrek. "The Teaching of Secondary Mathematics in Holland." *Mathematics Teacher*, Vol. XIX, 1926, pp. 329-42.

[4] W. Lietzmann. *Methodik des mathematischen Unterrichts*, Vol. II, pt. 1, pp. 334-39. Leipzig, Quelle and Meyer, 1926.

the school is still considered as elementary. This advanced work is not compulsory.

The secondary schools are the *Gymnasium* (with a six-year course, the ages of the pupils ranging from 12 to 17, 18, or 19) and the *Hoogere Burgher Schole* (Higher Burgher School), usually denoted as H.B.S. In the former the classical languages (both Latin and Green) predominate, whereas the latter is nonclassical. A school like the German *Realgymnasium* with Latin but without Greek does not exist as yet in Holland.

In the first four years (forms) of the *Gymnasium* the instruction is the same for all pupils, but the fifth and sixth forms are divided into two sections, A and B. In these sections many lessons are given in common, but while in section A the classical studies predominate the pupils in section B apply themselves especially to mathematics and science. The H.B.S. has usually a five-year course, although some have only the first three classes. In all these schools mathematics and the sciences are emphasized. While the *Gymnasium* is primarily a preparatory school for the universities, the H.B.S. aims rather to supply an all-round education; and to prepare for the higher technical and medical schools, and for various other special lines of work. As a rule the schools in Holland, secondary as well as elementary, are coeducational; that is, the schools for boys are open to girls; but there are also schools for girls only. In Holland there are also many other secondary schools known as lyceums (*lycea*).

A lyceum is not, as in Germany, a school for girls; it is only a combination of a H.B.S. and a *Gymnasium*. The first two years are the same for all pupils, a bifurcation taking place at the beginning of the third school year. As a change in the distribution of the lessons over the different years is permitted by law, this combination is possible. It is evident, therefore, that the Dutch lyceums are comparable with the *Reform-Anstalten* in Germany. It should also be understood that the lyceum is not required by law; at present such a school is merely permitted, although it is probable that before long the legality of the lyceum will be established.

Holland has three state universities (at Leyden, Utrecht, and Groningen), one municipal university (in Amsterdam) and two denominational universities (one protestant in Amsterdam and one catholic at Nimeguen). As a rule a university has five faculties (departments): theology, law, philology, medicine, and mathe-

matics-science. Besides these five, some universities have a sixth department (veterinary surgery in Utrecht, and commercial studies in the municipal university of Amsterdam), while in others (the two denominational ones) there are no medical and scientific departments. In addition to the universities there are four academies —a technical one at Delft, a commercial one at Rotterdam, an agricultural one at Wageningen, and a recently founded catholic commercial academy at Tilburg.

II. Development of Mathematical Teaching in Holland Since 1910

Various Phases of Report. Proceeding to the proper subject of this report, this discussion will consider: (1) the mathematical teaching in elementary education, (2) the mathematical work in secondary schools, (3) certain special topics and methods, and (4) mathematics in the universities and academies.

Elementary Education. As to the teaching of mathematics (arithmetic) in elementary schools there have been no important changes. Both subject matter and methods are still the same as they were about the year 1910, and this is also true for the more advanced (M.U.L.O.) instruction, which includes geometry and algebra, apart from a slight change in the final examination.

Secondary Education. Of much more importance are the changes that have taken place in secondary mathematics, especially in that of the *Gymnasium*, the pupils being from 12 to 17, 18, or 19 years of age. Up to 1919 the subject matter was prescribed for these schools in the Royal Decree of June 21, 1887; in a new Royal Decree of June 7, 1919, however, changes of relatively great consequence took place. For a clearer understanding of these changes the standard courses of 1887 and 1919 are here arranged in parallel columns:

OLD COURSE	PRESENT COURSE
Arithmetic and algebra in the first four forms. The elementary operations with integral and fractional numbers and algebraic expressions, divisibility of numbers, the metric system, proportions, linear equations with one unknown, simultaneous linear equations, surds, fractional and negative exponents; in the fifth and sixth forms, quadratics and recapitu-	Arithmetic and algebra in the first four forms. The elementary operations with integral and fractional numbers and algebraic expressions, divisibility of numbers, proportions, linear equations with one unknown, simultaneous linear equations, surds, fractional and negative exponents, easy quadratics, *computing with logarithms, graphs;* in the fifth and sixth

OLD COURSE (*continued*)

lation of surds, fractional and negative exponents.

Geometry in the first four forms, plane geometry. In addition to the above, in the fifth form and in the sixth form of the B section three lessons a week are required in arithmetic series, logarithms, in determinate linear equations, plane and spherical trigonometry, elements of the theory of coördinates in a plane, recapitulation and application to problems.

PRESENT COURSE (*continued*)

forms, *more detailed treatment of quadratics, recapitulation of algebra.*

Geometry in the first four forms, plane geometry, *first elements of trigonometry with application to the right triangle;* in the fifth and sixth forms solid geometry and a recapitulation of plane geometry. In addition to the above, in the fifth and sixth forms of the B section lessons are required in arithmetic *and geometric* series, logarithms, *plane trigonometry, plane analytic geometry including conic sections, elements of the calculus,* recapitulation and application to problems.

It must be said, however, that some teachers go much farther (particularly in the B section) than is required here, their work including an introduction to the binomial theorem, the solution of cubic equations, and Gauss's geometric representation of complex numbers in relation to De Moivre's theorem and the solution of binomial equations.

In the case of the H.B.S., the statement is different. Strange as it may seem, the subject matter for these schools was not legally decreed at all; it was only custom and the final examination requirements that determined what should be taught. The Royal Decree of June 16, 1920, gave the first official directions in this respect, as follows:

FORM I

Arithmetic: Properties of elementary operations. Divisibility, highest common factor and lowest common multiple, common and decimal fractions, problems, proportions.

Algebra: Elementary operations with integral algebraic expressions. Special products and quotients, factoring, linear equations with one unknown.

Geometry: Elements up to proportionality of segments of lines.

FORM II

Arithmetic: Proportions continued, square root, elementary notions on approximations.

Algebra: Easy cases of H.C.F. and L.C.M., fractions, linear equations continued, the same with more unknowns, surds (only those reductions that are applied in geometry).

Geometry: Elements continued up to and including the circle.

Form III

Arithmetic and Algebra: Fractional and negative exponents, logarithms, series, compound interest, quadratics, and related equations of a higher degree with one or more unknowns, graphs.

Trigonometry: Functions with applications involving the cases of a single angle.

Geometry: Elements following the circle, plane geometry being completed.

Form IV

Algebra: Logarithmic and exponential equations, recapitulation.

Trigonometry continued.

Geometry: Solid geometry up to solids of revolution. Introduction to descriptive geometry.

Form V

Algebra: Recapitulation.

Trigonometry continued: Easy trigonometric equations. Recapitulation.

Geometry: Solid geometry continued. Descriptive geometry up to the sphere. Recapitulation.

The chief features of the changes are as follows: Plane geometry is neither reviewed nor taught at all in the fourth and fifth forms; it is not included in the final examination requirements. In algebra indeterminate equations have been dropped, and so have been more intricate trigonometric equations. Graphs are now included, but it is not said to what extent they should be taught.

Naturally in the H.B.S. the same options are offered as in the *Gymnasien;* there are progressive teachers who go farther than is officially required.

Special Topics, Methods, and Tendencies. It is a remarkable fact that the development of mathematical teaching has been so little influenced by tendencies in foreign countries. The changes that have taken place in Holland have often been introduced independently of other countries and have come many years later than in Germany and France, where the so-called Reform movement began about 1900. The work of Professor Felix Klein and his collaborators in Germany, the Meran curricula drafted in 1905, the International Commission on the Teaching of Mathematics founded at Rome in 1908 at the suggestion of Professor David Eugene Smith, all these events of so great moment were known only to a few Dutch teachers. Gradually the new ideas began to influence the practice of teaching; especially the new standard course for the *Gymnasium,* of which we spoke above, was a stimulus. We shall now consider some cardinal problems which have been of great importance in reorganization in every country.

The Function Concept and Graphs. These features which, so far as I know, were not found in our teaching in 1910, have gradually been introduced. At first only the linear and quadratic functions were treated; in after years this study was extended to others, such as $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = x^3$, $y = \frac{ax+b}{cx+d}$, $y = \frac{ax+bx+c}{dx+e}$, and $y = \frac{ax^2+bx+c}{dx^2+ex+f}$, as well as the exponential and logarithmic functions. That these subjects are studied to a greater or less extent in the *Gymnasien* at present appears clearly from the final examination papers now in use. In the H.B.S. this development is at present in its initial stage. The first textbooks in which the function concept imbued the entire teaching of algebra, were those of P. Wijdenes.[5] There has recently appeared another work by three authors, which differs considerably from current textbooks of algebra.[6]

The Calculus. The German reformers in the beginning of this century laid stress upon the elements of the calculus being a necessary complement to the function concept. They pointed out that mathematics is not rendered more difficult by the introduction of the elements of the so-called "higher mathematics." As early as 1903 two Dutch teachers, F. J. Vaes and C. A. Cikot, demanded insistently but in vain the introduction of the calculus in secondary mathematics. During many years the resistance, even of prominent teachers, was strong; nobody seemed to know what had happened abroad.[7] In 1919, however, a change took place, the Royal Decree of that year ordering this subject for the *Gymnasien*. As a result, elementary calculus is now taught in the highest classes of these schools throughout the country, although not always to the same extent. In all cases, however, polynomials, fractions, square roots, sines, and cosines are differentiated and the process of differentiation is applied to cases of maxima and minima. Less general are the derivatives of the exponential and logarithmic functions, higher differential coefficients, and Taylor's and Maclaurin's series. In most cases the course is completed by the elements of integrating

[5] His numerous works are published by P. Noordhoff at Groningen.

[6] L. Yntema, A. J. Drewes, and Th. B. Bloten. *Algebra voor Voorbereidend Hooger en Middelbaar Onderwijs*, Parts I-IV. J. B. Wolters, Groningen.

[7] In the Congress of Paris (1914) of the International Commission on the Teaching of Mathematics, Professor E. Beké of Budapest gave an extensive discussion of this question (*Publications du Comite Central*, 2^{me} *serie, fasc.* III, pp. 59-122). In many countries the elements of the calculus had already been introduced.

(application on finding areas and volumes).[8] In the H.B.S. the calculus has not been introduced; some teachers, however, apply its principles in treating the notions of rate and acceleration in mechanics.

Intuitive Geometry. With the exception of a few schools demonstrative geometry is taught without a preliminary intuitive course. Such a "propædeutic" introduction, as it is called in Germany, still meets with much opposition. It has been strongly advocated by W. Reindersma, who has written a textbook[9] in which demonstrative geometry is preceded by an intuitive course.

History of Mathematics. In many countries the claim has been made that the historical development of mathematics and biographies of great mathematicians should have a place in our teaching. As early as 1912 a treatise on this subject was written by M. Gebhardt for the German Branch of the International Commission on the Teaching of Mathematics. In Holland an interest in the subject is gradually awakening; textbooks with historical and biographical notes, surveys of the history of a certain branch, and portraits of mathematicians have come to be quite common.

Applied Mathematics. It is remarkable that practical applications of elementary mathematics have been entirely neglected in Holland. In many other countries not only the disciplinary but also the practical aims are in evidence, as in Germany and the United States, but not so in Holland.[10] Neither measuring and estimating in the classroom nor geodetic measurements and surveying are known here. In only one school, so far as I know, is a theodolite found. Similarly, there are no angle mirrors, measuring rods, and the like. The slide rule is also unknown in our mathematical teaching. All this explains the different appearance of a Dutch textbook on mathematics from that, for example, of an American textbook.

Universities and Academies. The range of instruction in the universities is determined by law, and each professor is appointed

[8] A remarkable little book on functions and elementary calculus, of which T. Percy Nunn's *Teaching of Algebra* has evidently been the prototype, is *Functies*, by J. Droste and W. F. de Groot. Two volumes, published by J. B. Wolters, Groningen.

[9] W. Reindersma, *Inleiding tot de Vlakke Meetkunde*, and *Beknopt Leerboek der Vlakke Meetkunde*. J. B. Wolters, Groningen.

[10] A survey of what had been done in this respect in different countries was presented by Professor David Eugene Smith to the Congress of Cambridge (1912), being a report of the International Commission on the Teaching of Mathematics (*Publications du Comité Central, 2me serie, fasc.* I, pp. 67-94).

for a special branch. No changes have taken place recently in this respect. Within certain bounds, however, the professor is free to select his subject matter, and so a change in the topics is possible whenever a professor's chair is taken by his successor. Such new topics have been, for instance, the theory of numbers in general, of irrational numbers, and of assemblages, and the history of mathematics. In the Technical Academy at Delft some changes should be mentioned. The requirements for mathematics at the first examination of the future engineers (the so-called propædeutic examination) have been considerably reduced, especially with respect to analytic and descriptive geometry. Although exercises in descriptive geometry have been given for many years, this has not been the case in analysis and analytic geometry, and only recently has this omission been remedied. The Agricultural Academy at Wageningen has recently (in 1918) been developed from the State High School for Agriculture, Horticulture, and Forestry, although even before that year mathematics had already secured a significant place in teaching. Even in 1913 a separate chair for pure mathematics was instituted. All students now have a compulsory course in the elements of plane analytic geometry (including conic sections) and the calculus. Those who have the degree of candidate may follow lectures on the theory of probability and mathematical statistics. Special courses for surveyors, including solid analytic geometry, descriptive geometry, spherical trigonometry, method of least squares, etc., have been established in this academy.

III. Recent Efforts for Improving Mathematical Teaching in Holland

Efforts to Improve Teachers and Teaching. Our report would be incomplete without mentioning the efforts that have been made to improve our secondary mathematics and the preparation of teachers. It has already been pointed out that the teaching in the *Gymnasien* is more in accord with modern tendencies than that in the H.B.S., although it is difficult to assert that this is the case in general, the results depending for a great part upon the personal ideas of the teacher. One thing, however, is certain, the examination papers of the H.B.S. are still old-fashioned, a matter of regret to many prominent teachers. That the want of improvement has been felt appears from the fact that a semiofficial committee was organized towards the close of 1925, the purpose being

to undertake a comprehensive study of the teaching of secondary mathematics and related branches, and to suggest improvements in the H.B.S. teaching. The committee consists of four members: Dr. H. J. E. Beth, chairman; Dr. E. J. Dijksterhuis, secretary; P. Cramer; and J. van Andel. It is usually denoted as "Committee-Beth." It has published a report[11] in which it proposes a new standard course. Evidently the bounds that have been set to this survey do not allow even a succinct review of it being given here; the reader may consult a paper on the report by the present writer.[12] Only some typical features may be mentioned. In general the proposals correspond to the claims that are heard in Germany, the United States, and other foreign countries; algebra teaching ought in the first place to promote functional thinking and graphs should be studied as well as the elements of the calculus. Complicated numerical computations are condemned, and so are the still rather frequent intricate exponential and logarithmic equations. In plane geometry an elementary synthetic treatment of the ellipse, parabola, and hyperbola is desired, and in solid geometry their genesis in relation with Dandelin's theorem. As to the desirability of an intuitive course preceding demonstrative geometry the committee takes no definite stand. It recommends as yet "the moderate Euclidean method." A striking fact is that the committee rejects applied mathematics altogether—a remarkable difference with modern tendencies abroad. The report has been followed by two others, in which the training of teachers is discussed.[13] The level of scientific studies in the Dutch universities ought to be a high one, as has always been the case. But the universities should pay more attention to the student's future field of activity as a teacher in a secondary school; therefore "elementary mathematics from a higher point of view," as Professor Felix Klein has called it, should be taught. In certain other respects there are also lacunæ; the philosophical as well as the historical aspect of mathematics is neglected in nearly

[11] *"Ontwerp van een leerplan voor het onderwijs in wiskunde, mechanica en kosmographie op de H. B. Scholen met vijfjarigen cursus." Bijvoegsel van het Nieuw Tijdschrift voor Wiskunde,* Vol. II, 1925-26, pp. 113-39. Sold also as a separate pamphlet, P. Noordhoff. Groningen.

[12] D. J. E. Schrek. *"Reformbestrebungen im mathematischen Unterricht an den Hollandischen Realanstalten." Zeitschrift für math. u. naturwiss.* pp. 361-64. *Unterricht,* 57 Jahrg, 1926.

[13] *"Beschouwingen over de universitaire opleiding tot leeraar in wis-en natuur kunde." Bijvoegsel van het Nieuw Tijdschrift voor Wiskunde,* Vol. II, 1925-26, pp. 81-95. *"Nadere beschouwingen over de opleiding tot leeraar in wis-en natuurkundige vakken." Ibid.,* pp. 146-57. Published separately by P. Noordhoff, Groningen.

all our universities. It is Dr. Dijksterhuis, a historian of note himself, who advances a claim here, and undoubtedly he is right.

In conclusion I wish to thank all those who have made my task easy in giving me information and providing me with data.

HUNGARY

By PROFESSOR CHARLES GOLDZIHER

Budapest

General Considerations. Since the close of the World War, Hungary has hoped to counteract the effect of her material impoverishment and territorial losses by the raising of her intellectual and cultural standards. Accordingly, far-reaching changes in public education have taken place, these being particularly concerned with the enlargement of the curriculum and with the enrichment of the program of schools of every type. These changes are of great importance in the consideration of the reform in the teaching of mathematics since the adoption of the new syllabi and the instructions to the teachers, together with the reorganization and development of certain types of schools, presented a favorable opportunity for the consideration of the recommendations of the International Commission on the Teaching of Mathematics. The reform in the teaching of mathematics had begun in Hungary before the War [1] and it is a significant consequence of the sound practice of the Hungarian Committee which has been at work since 1906 that the new objectives could be realized by a continuous transition from the former situation.

Before entering into the details, let us consider certain characteristic traits of the teaching of mathematics in Hungary:

1. The principal issues of the Hungarian reform movement have been listed on page 25 of our report in the *Abhandlungen,* which

[1] See *Abhandlungen über die Reform des mathematischen Unterrichts in Ungarn, Leipzig und Berlin,* edited by Beke and Nikola. Teubner, 1911.

For the former situation in the teaching of mathematics, see the eight special reports of the Hungarian Subcommission.

See also the Hungarian titles in the *Bibliography of the Teaching of Mathematics 1900-12* by David Eugene Smith and Chas. Goldziher (U. S. Bureau of Education, Bulletin, 1912, No. 29).

For the situation in 1912-14, the reader is referred to "Mathematical Curricula in Foreign Countries" by J. C. Brown in the *Report of the National Committee on Mathematical Requirements,* 1923, in which the reports sent by the various countries to the International Commission in its Cambridge congress, 1912, are summarized.

has been mentioned previously. These questions conform in general to the principal desiderata of the IMUK report. The fact must be emphasized, however, that we laid especial stress on the improvement of the elementary grades, the lower classes of the high school, and the secondary school so that the new ideas might permeate the entire course as soon as possible.[2] An important result of this effort, for example, is to be found in the new and detailed Instructions (1927) which supplement the usual syllabus for the secondary schools. The mathematical part of this sets up a standard for the teacher, telling how the reform program may be put into effect in the lower grades.

2. A monthly journal has had great influence on the proficiency of school work in mathematics in Hungary. Ever since 1893, it has inspired the pupils in all the high schools of the country to common work in mathematics.[3] In 1924, the journal was reorganized under the title: *Középiskolai Mathematikai És Physikai Lapok* (High School Papers in Mathematics and Physics) and it is interesting to note that problems which concern the geometrical theory of functions, graphic work, applications of the elements of the calculus, and descriptive geometry have the greatest appeal to the students. A special section of this journal is devoted to pupils of the middle grade; this is headed "Exercises." In it are published the original drafts of the best solutions to problems, together with the names of other contestants. The numbers of the journal also contain articles on historical and other special topics. The best university and technical school students grow up in the workshop of this magazine, which is an influential leader in the modern ways.

3. Since 1896, the Hungarian Mathematical and Physical Society has set annual written examinations in mathematics for the graduates of the high schools. (Since 1919 there have been separate examinations in physics also.) And since 1924, similar examinations in all subjects have been set by the Ministry of Education for pupils selected from the high schools of the entire country.

4. The system of summer schools for teachers (vacation lectures) has been cultivated in Hungary[4] and in 1912 the high school teachers held a congress at which reports were presented and in

[2] For this see our study in the *Zeitschrift für math. und naturwiss. Unterricht*, 1908, pp. 289-309. This periodical will later be referred to as the *Z. M. N. U.*

[3] See *Z. M. N. U.*, 1910, p. 519.

[4] See *Z. M. N. U.*, 1914, No. 3, on lectures treating the practical relations of the teaching of mathematics.

which discussions concerning the reform movement in mathematics were held.[5]

Significant Changes Since 1910. In the next part of this report, we shall enumerate only those types of schools in which the teaching of mathematics has undergone significant changes since 1910. We shall mention only the following types:

1. Elementary or primary schools (*Volksschule*), pupils aged 6 to 12 or 6 to 10 if they continue their work in a high or in a secondary school. These schools have greatly increased in numbers, especially in rural districts. The arithmetic and geometry taught here has developed in a modern way but these changes have been independent of the reforms in other schools.

2. Agricultural, industrial, and commercial schools and training courses with many ramifications. These show some improvements in their organization and in the adaptation of the teaching of mathematics to special practical requirements.

3. A Business College (Economical University) organized in 1920 in Budapest. The work in mathematics is of modest proportions, but in the Faculties of Commerce and Insurance advanced courses are given in the mathematics of economics, insurance, and statistics, and in the theory of probability. Advanced courses in the mathematics of business and insurance are also given under the Faculty of Economics at the *Polytechnicum* in Budapest which was established in 1914.

Details of Organization. The discussion of the more detailed part of the curriculum may be grouped under four heads as follows: I. New Curricula for the Reorganized Schools; II. New Curricula for Schools of the Old Type, III. New Curricula for Augmented School Courses, and IV. Preparation of Teachers.

I. New Curricula for the Reorganized Schools. Mention should be made of the reorganized high schools for students of ages 10 to 18. The high schools for boys which formerly offered two courses have offered three since 1924. These courses are the following:

1. The *Gymnasium*, emphasis on a humanistic education. Greek studied in classes above the fifth.

2. The *Realgymnasium*, emphasis on Latin and modern languages (French, English, or Italian).[6]

[5] See *Z. M. N. U.*, 1913, p. 571, and *Pädagogisches Archiv*, 1912, pp. 645-51.

[6] Hungarian and German are taught in every high school.

3. The *Realschule,* emphasis on modern language, mathematics and the natural sciences.

These three schools are on a par in regard to admission to the university or to the polytechnic institutes.

Since 1926, the high schools for girls have the following threefold organization:

1. The *Gymnasium,* emphasis on humanistic education, Latin being offered from the third class instead of Greek and French.
2. The *Lyceum,* emphasis on modern languages and æsthetics.
3. The *Collegium,* emphasis on the economic and industrial branches that are of importance for girls.

Like the boys' schools, the first two of these schools prepare students for advanced work. The third, however, prepares candidates for special professional or training colleges only.

The Reform Movement in Mathematics. The reform movement in mathematics had the high schools as its main objective, and accordingly the syllabi reflect its influence. The details that follow are based on the work of the high schools for boys, these changes having been made by reducing, transposing, and fusing the old subject matter:

1. The introduction of graphic elements in the beginning work in arithmetic, and the introduction of the function concept in the beginning of algebra in the third class, the evolutional development of these elements in all subsequent classes.
2. The enriching of the formal work of every class by the practical elements of real measuring and numerical computations, approximate computations being given in the *Gymnasium* from the fifth class only.
3. The introduction of the elements of analytic geometry and calculus in the seventh class in a modest measure but one which is necessary for a general education.
4. The enlargement of solid geometry, building the work of the higher classes on its descriptive elements and emphasizing the development of "spatial intuition" throughout.
5. The methods of applied mathematics are studied and the uses of mathematics in physics, engineering, and economics are made the core of practice exercises.
6. The fusion of algebra and geometry in their empirical (or experimental) aspects in the lower grades and in their more formal

aspects in the higher ones. Geometry was formerly a separate subject in the first four classes, being given in its empirical or propædeutic form, but to-day it appears as a separate subject only in the *Realschule*.

The further extension of this process in the future is likely to be in the direction of the fusing of plane and solid geometry together with the wider development of the "laboratory method." [7]

The amount of mathematics offered is naturally the greatest in the *Realschule* where the geometry of all classes is more detailed and where in the fifth through the eighth grades a separate course in descriptive geometry is given. The analytic geometry of this school is more comprehensive, and at the close of the course work on the theory of complex numbers is also given.

No calculus is offered in the high schools for girls. The practical elements of mathematics are emphasized and the formal side of the teaching is simplified.

The following statement of the hours per week allotted to mathematics gives an idea of the extent of the work in the different types of schools.

Class	Boys' Schools				Girls' Schools		
	Gymnasium	*Realgymnasium*	*Realschule* Math.	*Realschule* Geom.	*Gymnasium*	*Lyceum*	*Collegium*
I	6	6	4	3	4	4	3
II	4	4	5	2	4	4	3
III	4	4	3	2	3	3	3
IV	4	4	4	2	3	3	2
V	3	3	4	2	3	3	2
VI	3	3	4	2	3	3	2
VII	3	3	3	2	3	3	2
VIII	2	3	3	2	2	3	2
Total	29	30	30	17	25	26	19

II. New Curricula for Schools of the Old Type. To this group of schools belong the four-year secondary schools for pupils aged 10 to 14. These schools, whose name in Hungarian is *polgári iskola* (citizen schools), were instituted in 1868. They ordinarily prepare for professional or training schools, and they have an important rôle because of their emphasis on practical things. They are

[7] See our paper in *School Science and Mathematics*, 1908, pp. 753-57.

chiefly found in towns where there are no high schools and they are of great national importance. Such schools are urgently needed for girls who go no further in their school work. The new syllabus of 1918 shows valuable improvements in mathematical studies, these being in harmony with the practical character of the school. In this syllabus, unnecessary and formal details are omitted together with antiquated subject matter. Instead, the *Sachunterricht* with a laboratory method has been introduced. Empirical and intuitive methods appear in geometry; numerical and graphical computation is a leading factor in arithmetic; and applications to statistics, business, industry, and agriculture are given in the very beginning of the work in algebra. A significant item is the fusion of all mathematics in one subject; even commercial arithmetic and bookkeeping are not excepted. The time allotment is 4, 4, 4, 2, or 14 hours for the boys, and 4, 3, 2, 2, or 11 hours for the girls. But the execution of these plans depends in the first instance on an adequate and specialized preparation of the teachers for these particular types of schools.

III. New Curricula for Augmented School Courses. Since 1920, the higher commercial schools which formerly had a three-year course, have had a four-year course for students from the ages of 14 to 18. The syllabus of 1927 gives a better distribution of the work and enlarges the content of the commercial arithmetic. It also gives practical suggestions for the teaching of mathematics but the time allotted to formal work in algebra and geometry is too short, and political arithmetic (i.e., the mathematical principles of finance) is not separated from algebra. The new ideas have provided certain ornaments for the course without affecting the method of teaching. The time allowance for mathematics and political arithmetic is 3, 2, 2, or 7 hours, and for commercial arithmetic 4, 2, 3, 3, or 12 hours.

The normal school course for elementary school teachers was increased from four to five years in 1923, the students being from 14 to 19 years old. These institutions are equipped with elementary schools for demonstration work and practice teaching. The syllabus of 1925 offers some improvement over the earlier, very scanty course in mathematics by the introduction of many practical applications, the concept of a function and the use of graphs but the formal extension of the course, which does not include trigonometry and logarithms, is not satisfactory. Great stress is laid on the details

of the pedagogy of the elementary schools subjects in the fifth class. The time allotted to mathematics is 3, 3, 2, 2, 2, or 12 hours.

IV. The Preparation of Teachers. The preparation of teachers shows no great change in its mathematical content since 1910, but it should be stated that all training institutions conform to the changes in the organization of special schools. The courses for the teachers of the elementary schools have been mentioned above. Teachers of the commercial schools are given special preparation in the Business College at Budapest. There are no institutions devoted specifically to the preparation of teachers for the other professional schools.

The teachers of secondary schools receive their special training at the two government teachers colleges, one for men and one for women. The course is for three years, and students are admitted at the age of 18. These colleges have excellent practice schools. Since 1920 there has been a separate faculty for mathematics, physics, and chemistry, with two professorships in mathematics, one of them being in applied and business mathematics. The treatment of pedagogical questions is emphasized, and formal training in mathematics conforms to that of the first two years in the university. These colleges, with their precious traditions of sixty years' standing, have an autonomous organization and a well-differentiated course of studies adequate to the needs of a limited national extraction. It is expected that these colleges will be annexed to one of the universities as a four-year college, a thing which actually was done in October, 1928, at the University of Tzeged.

High school teachers receive their theoretical training in the universities simultaneously with their pedagogical training in the normal courses. They then do their practice teaching in independent high schools. The training of high school teachers is thus in the closest relation to the scientific work at the university. In most European countries it is an open question whether the professional training of teachers should be given in autonomous colleges or in a normal course in a university. The old normal school system in Hungary dating from 1870 had negligible results in the training of high school teachers, for the only one of effective worth was the *Übungsgymnasium* in Budapest founded as a model school by the great educator, M. Kármán, in 1872.[8] It is to be hoped

[8] See the report of P. Szabó: *Der Unterricht der Mathematik am Übungsgymnasium,* Hungarian Subcommission of the IMUK conference, 1912.

that the current reorganization of the normal courses mentioned above will bring greater results in the future.

In the *Abhandlungen*, pp. 126-41, there appeared a paper by E. Beke in which the defects in the modern training of high school teachers of mathematics are explicitly formulated. Certain of these demands have already been realized, but it is desirable that applied mathematics with its special branches be included in the mathematical course in the universities and that this should be allotted a separate section of the examinations. This lack affects the mathematical lectures at the *Polytechnicum* also, where laboratory work in applied mathematics should be made an important part of the studies in mathematics. Institutions in America, England, and Germany have been investigated in these particular details. It is only by the establishing of such institutions that the preparation of high school teachers of mathematics can be improved in any considerable degree.

ITALY

By PROFESSOR FEDERIGO ENRIQUES
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Gentile's Reform. The great factor which has dominated the recent changes in education in Italy is Gentile's Reform of 1923. Gentile, the idealistic philosopher, who was appointed minister of education by the Fascist government, has accomplished far-reaching changes in schools of all types. The general bearing of his Reform as it affects the secondary schools is as follows:

1. Both public and private schools, notably the parochial ones, are put on a common footing in regard to the newly instituted state examination for admission to the universities.

2. The schools are expected to build attitudes rather than to furnish facts whose justification is utility or else specific preparation for more advanced studies.

3. The organization of the curriculum is based upon a literary, historic, and philosophic point of view. Latin appears in the courses given in all types of schools; that is to say, not only in the *ginnasio* and in the classical *liceo*,[1] but also in the newly created scientific *liceo* which prepares for the scientific course in the universities and which replaces the technical institute and the modern *liceo*.[2]

4. The syllabus and the directions given to the teachers are planned to foster a method of instruction which shall appeal to the initiative and the interests of the pupil to the greatest degree possible. Accordingly, the syllabi for the state examinations allow a certain amount of choice. For example, in preparation for the examination in philosophy, the student is asked to discuss the thought of classical authors chosen from an extensive list, and so on.

[1] According to J. C. Brown's investigation as given in the *Report of the National Committee on Mathematical Requirements* (1923), the compulsory school age in Italy is from the sixth to the twelfth years and a six-year elementary school is provided to care for children whose school training is to last no longer. Entrance examinations for the secondary school are given at the end of the fourth school year. This admits the student to the five-year *ginnasio* from which, after another examination, he enters the three-year *liceo*.—Editor.

[2] These schools formerly paralleled the *ginnasio* and the *liceo*. No Latin was taught and the course was a year shorter than the classical one.

5. There is a tendency to lessen the number of instructors teaching each class. Thus in the *liceo,* one teacher is given the work in history and philosophy, another has mathematics and physics, and so on.

Results of the Reform. The Reform has led to a reduction in the number of hours allotted to the sciences. This has caused considerable difficulty in the case of the teachers of mathematics and physics who, for the most part, are new to the teaching of one or other of the two subjects. Furthermore, as a result of the syllabi for the state examinations, the scope of the subject matter to be taught has not been diminished, nor can it probably be diminished.

Preparation for the University. The arrangement of the curriculum reflects the spirit of the Reform. Accordingly, we will pause a moment to consider the course which leads to the diploma of the classical *liceo* which is the usual way of preparing for the university.

There are two different sections of this course. Section A is concerned with algebra and trigonometry (equations of the first and second degrees, exponential equations and logarithms, periodic functions, and applications). The purpose of the section is to test the student's ability to use fundamental formulas and the examination consists of several simple exercises to be solved under the direction of the examiner.

Section B, which is concerned especially with geometry, is quite different. The purpose of this section is the testing of the intelligence of the candidate and of his ability to comprehend the rigorous, deductive systematizing of a theorem. The propositions which it contains correspond in the main to Euclid's *Elements:* proportion and similar plane figures, irrational numbers, the measurement of the circle, and the elements of the geometry of space. The candidates are allowed a choice from three topics of sterometry: equivalence and congruence of polyhedra, surfaces and volumes of curved solids, and similarity in space with its special cases. (The elements of algebra and of plane geometry, that is the congruency and equivalence of polygons, form a part of the entrance examination of the *liceo.* This examination is of the same type as the state examination and follows the same regulations.)

The mathematics examinations of the other schools are in two sections also, but in the case of the scientific *liceo,* the mathematics program is more extensive. It includes analytics and the elementary concepts of the infinitesimal calculus, that is, the work which was

formerly given in the modern *liceo*. This has been outlined by Castelnuovo.[3]

General Spirit of the Teaching of Mathematics in Italy. It is evident that the general spirit of the teaching of mathematics in Italy is in accord with the spirit of classical education. It is still under the influence exerted by Betti and Brioschi who, upwards of a half century ago, restored the *Elements* of Euclid to our schools, which prior to that time had been using books on the order of Legendre. It might have been expected that there would have been a more practical development of algebra, with less attention paid to theory and more attention to problems drawn from physics. A movement in this direction may result from this Reform eventually, but at present this is not evident in our textbooks, except in those designed for the scientific *liceo*. Here the innovation was to be noted before the Reform both in the *Nozioni* by Amaldi-Enriques, noted above, and in the *Algebra* of M. Marcolongo.

New Books. Several new books on elementary mathematics have appeared recently in addition to the books on algebra and on geometry written by Pincherle and Enriques respectively, printed by Zanichelli and Bologna, and other well-known works, such as the elementary geometry by De Franchis printed by Sandron in Palermo.

I will mention two of these series in particular. The first is edited by Marcolongo of the University of Naples and Niccoletti of the University of Pisa. Their work is printed by Perrella in Naples. The second series is edited by Severi of the University of Rome and is printed by Vallecchi in Florence. The first series, begun before the Reform and so witnessing the modification or the suppression of the institutes,[4] already includes several volumes: the *Algebra* of Marcolongo mentioned above, another *Algebra* by Sansone, a *Trigonometry* and a *Geometry* based on vectors by Marcolongo and Burali-Forti, and the *Geometry* written by Rosati and Benetti. The second series includes an *Algebra* by Bagnera and a *Geometry* by Severi. Almost all of the authors now belong to the universities, although they have in general taught in the secondary schools at one time or another. This is an indication of the interest now current in questions of education.

[3] See *Nozioni di Matematiche* by Amaldi-Enriques, printed by Zanichelli, Bologna.

[4] These were the *instituto tecnico* and the *instituto nautico* offering four- and three-year courses respectively, and giving the last years of the non-Latin secondary school course.

The Teaching of Geometry. I must not pause to discuss the criteria on which these books are arranged in detail, lest this discussion should become burdensome. I shall, however, say a few words on the most discussed questions in the teaching of geometry. The methods of teaching this subject and, above all, the introduction of its concepts have been the subject of a number of studies, some from the pedagogical point of view, others from that of logical criticism. The influence of the latter is especially evident in the Italian textbooks. This is apparent at the very beginning in the care with which the principal properties of the geometry of position are explained, the arrangement of points on a line, segments, and so on. I shall not say more of the method by which the exigencies of intuition and rigor are combined, for this has been treated in detail by Enriques and Amaldi.

One of the most important differences is that which pertains to the definition of equal or similar figures. Euclid's method is well known. He considers the concept of congruent figures as a postulate. Equality is always recognized as equality of size—equality of segments or of angles, of surfaces or of solids. There is no general definition of equality of form, but equality of form of two triangles is expressed by saying that they have equal sides and angles (Book I, 4, 8, 25). Later Euclid (Book III) defines equal circles as circles that have equal diameters. A general definition of similar figures appears at the beginning of Book VI, where Euclid says that two rectilinear figures are similar if they have equal angles and if the sides enclosing those angles are in proportion. In testing the equality of two triangles, Euclid twice uses superposition by motion (Book I, 4, and 8) which he employs in no other place and which he has not included in his postulates. This use of motion is not really necessary except in the first case where the equality of triangles having two sides and the included angle equal is established. A proposition so established resembles a postulate rather than a theorem. Hilbert's criticism of this point has made the Euclidean method the more rigorous, and it is this rigorous method which Enriques and Amaldi adopted and developed in their treatment, clarifying it by intuitive considerations in which motion is employed to a great extent.

Other authors prefer to introduce motion, recognized in all its generality, to define the equality or the congruence of two given figures. This is done glibly in the French texts with no concern

about the analysis of the significance of motion. In Italy the same plan is followed in the secondary schools of a lower grade, as appears in the abridged edition of Enriques and Amaldi, for example. Although due attention is given to the elementary work in the *liceo-ginnasio,* a need is felt for an analysis which shall consider motion as the correspondence between planes or between two solids and by which one may also attempt to derive the equality of angles from that of segments. Veronese and Ingrami were pioneers in this direction, but their work was too abstract. Several authors have followed their lead, De Franchis among the number, and more recently Rosati and Benedetti, and Severi. In the books written by De Franchis and by Severi, the discussion is limited to the logical justification of the terminology of motion.

At the same time, this discussion may lead to the general concept of similar figures in the manner of projective geometry; similarity is defined as a relation in which to every segment there corresponds another segment and in which the size of angles is invariant. Naturally, it is necessary that the teacher direct the attention of his students to the implications of this definition; otherwise some one will say that two rectangles are always similar since their angles are equal.

Preparation of Teachers for Secondary Schools. A few words should be added on the subject of the preparation of teachers for the secondary schools. Gentile's reform of the universities established certain principles for their development. Among these were: free election of studies, the state examination, and the independence of the university.

The state examinations for teachers of secondary schools are competitive. These examinations are also a necessary prerequisite for teaching in the private schools. The syllabus for the examinations, at least for the candidates for positions in the *liceo-ginnasio,* combines mathematics and physics. In mathematics, the outline corresponds for the most part to that given in the *Questioni riguardanti le Matematiche elementari* compiled by F. Enriques, of which only the part relating to geometry has been translated into German (printed by Teubner), the translation being made from the second Italian edition.

Preparation for these examinations is offered by a course given by the science faculties called "Complementary Mathematics." This is taken by the candidate for a certificate in mathematics and

physics combined, an innovation introduced by the Minister of Education, Corbino, prior to the reform made by Gentile. It is likely that the autonomy of the universities will allow the preparation of prospective teachers to develop in many ways. For example, the publication of a series of classics has been begun by the University School of the History of Science, which is associated with the new National Institute of the History of Science, and which is near the University of Rome. These classics are translated into Italian with critical historical notes. The first volumes were printed by Stock in Rome, but the publication is now being made by Zanichelli in Bologna. The volumes of this collection already issued are:

Euclid and His Ancient and Modern Critics (Vol. I, Books I-IV), by several collaborators under the editorship of F. Enriques.

Newton's "Principia," with Notes on the History of Mechanics, by Enriques and Forti.

The Method of Archimedes and the Origin of Infinitesimal Calculus in Antiquity, by E. Rufini. (This is especially intended for the training of teachers.)

Dedekind's Memoirs on the Axioms of Arithmetic, with Historical and Critical Notes, by Oscar Zariski.

Several of these works have already been made the texts of a course at the University of Rome especially designed for future teachers.

Conclusion. I do not know whether these comments are a sufficient answer to the question at hand. It must be realized that these recent reforms may result in changes which are not even dreamed of as yet.

JAPAN

By PROFESSOR YAYOTARO ABE

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Introduction. As the Report on the Teaching of Mathematics in Japan, which was prepared in 1912 by the Japanese Sub-Commission of the International Commission on the Teaching of Mathematics, set forth in detail the conditions and methods of teaching mathematics covering all kinds and grades of schools in the Empire at that time, the present report will deal only with important changes of mathematical instruction which have since developed. On account of the limited space those concerning chiefly primary and secondary education will be treated.

I. The Elementary School

Nature. The elementary schools are divided into an ordinary elementary school course which is six years in length and a higher elementary school which requires two or three years. Since the ordinary elementary school course is compulsory, a child who has reached school age (six years) must be sent to an ordinary elementary school to complete the course. After completing the course, some children may go into actual life, while others may enter a higher elementary school or a secondary school for more advanced education.

As the regulations provide, the teaching of arithmetic in the elementary school aims at making children properly versed in ordinary computations and in the necessary knowledge for actual life and in rendering their thinking sound and accurate. The weekly arrangement of its teaching hours for each year is as follows:

Subject	Ordinary Elementary School						Higher Elementary School		
	Year I	Year II	Year III	Year IV	Year V	Year VI	Year I	Year II	Year III
Arithmetic	5	5	6	6	4	4	4	4	4

In selection and arrangement of the subject matter of arithmetical instruction in the elementary school, a great deal of attention was given to its logical development in former days; but the recent and general trend in teaching is to consider it more important to look after the natural development of the mental ability of children by emphasizing the interest and the practical benefits.

The arrangement of the teaching materials of arithmetic for each year of the elementary school is as follows:

I. Ordinary Elementary School

1. *First year*—Numeration and notation of numbers up to 100; simple computations; applied problems.
2. *Second year*—Numeration and notation of numbers up to 1,000; simple computations; monetary units (yen, sen); length (metre, centimeter, millimeter); time (day, hour); applied problems.
3. *Third year*—Computations of integers; monetary units (yen, sen, rin); length (km., m., cm., mm.); capacity (l., dl.); weight (g., kg.); time (day, hour, minute, week, year, month); applied problems.
4. *Fourth year*—Computations of integers; numeration and notation of decimal fractions; simple computations; length; area (square meter, square centimeter, are, hectare); volume (cubic meter, cubic centimeter, cubic decimeter); weight (gramme, kilogramme, ton); time (year, month, day, hour, minute, second, week); square; rectangle; cube; rectangular parallelopiped; angle; directions; various kinds of applied problems.
5. *Fifth year*—Computations of integers, common and decimal fractions; supplement of units of metric system (nautical mile); outline of Shaku-kan system; particular properties and areas of rectangle, square, triangle, polygon, parallelogram, trapezoid and circle; particular properties and volumes of rectangular parallelopiped, cube, parallelopiped, prism, circular cylinder and sphere; currencies; computations of angles and time; applied problems. (Soroban-calculations may be added.)
6. *Sixth year*—Proportion; percentage; similar figures; scaled drawing; graphs; applied problems. (Soroban-calculations may be added.)

II. Higher Elementary School

1. *First year*—Integers; decimal fractions; algebraic computations of numbers; geometrical figures; units of weights and measures (micron, millilitre, milligramme, carat); soroban-calculations; applied problems.
2. *Second year*—Proportion; percentage; algebraic computations of numbers; geometrical figures; soroban-calculations; applied problems.
3. *Third year*—Supplementary treatment of those subjects already given in preceding years. (An outline of everyday bookkeeping may be given in order to meet the local requirements.)

Materials of Instruction. The materials of arithmetical instruction in the elementary school have gone through several changes, the most outstanding one being that which was brought out by the revision of standards of weights and measures. The Japanese standards of weights and measures consisted of the Shakukan system, the metric system and the yard-pound system which were all in joint use, but after thorough investigation of many years these compound systems were abolished and the metric system was made the only legal system after it was enacted in April of 1921 and put in force on July 1, 1924, with a reservation for the use of the old systems for a certain period of time.

Owing to such a change of standards, the subject matter of arithmetical instruction in the elementary school is now based on the metric system, and in consequence it has been made possible to appropriate for many other useful subjects such time and labor as was wasted heretofore for treating compound numbers; additions of numerous subjects concerning algebraic computations of numbers, graphical representation, and geometric figures being the notable benefits thereof.

The formulation of arithmetical facts and the solution of problems by using letters are treated even in the elementary school; in the higher elementary school are taught linear equations with one unknown quantity, negative numbers, integral expressions, linear simultaneous equations and fractions in the first year, and quadratic equations in the second year, the equations treated here being chiefly those with numerical coefficients. Graphs are treated mainly for intuitive representation of statistical data and functional relations which are simple and practical. The teaching of geometrical figures is limited to those which are necessary to common life and in general it is taught in an intuitive way. A prevailing tendency is to apply simple circular functions to certain computations concerning angles.

Soroban-calculation is the peculiar computation by means of a soroban which can be manipulated with considerable amount of simplicity and quickness. A remarkable merit of this calculation is that it lays down the foundation necessary for mental arithmetic. In many ordinary elementary schools soroban-calculation is taught, and it is a compulsory course in higher elementary schools because it was experimentally ascertained that children of higher elementary school grade can make the best use of it.

The Textbooks. The textbooks are compiled and published by the Department of Education and are in general use in the Empire; but as they merely show the standard of subject matter and required progress, selection and treatment of the materials are left to the judgment of teachers so as to adapt the actual teaching to the natural development of children and the local circumstances to the most desirable extent.

Since 1909 the textbooks have been revised three times, the last revision being started in 1925. This is mainly due to the change in the standards of weights and measures. Of the last revision those of the ordinary elementary school and of the first year of higher elementary school have already been finished and the rest will be completed shortly.

Method of Teaching. Several improvements have been made in the method of teaching as follows:

1. In the hope that children can be made to take an active attitude in learning arithmetic, much attention is given in the beginning to their amusement and interest, and careful consideration is given later to the treatment of those problems, the solutions of which children would obtain spontaneously. In recent years much emphasis and attention have been given to the construction of problems by the children themselves.

2. In actual teaching the mental ability of each child is taken into full consideration and although the fundamental subjects are given in common, applied problems are advanced in proportion to the individual capacity of the children. In some schools the Dalton plan is tried for experimental purposes, but some prefer the Winnetka plan to the Dalton plan.

3. It is acknowledged to be more profitable for children to find and construct by themselves the fundamental rules and methods of arithmetic than for teachers to explain and teach them to the children from the start; consequently, it is deemed very important to develop the subject matter by treating such problems as suit the natural development of the children.

4. It has long been the practice in Japan to make use of experimental work in teaching arithmetic of a preliminary nature, and this practice has been much encouraged since Perry's movement made a wide impression. Many schools are equipped with all kinds of scales, rules, and measures, and even with an arithmetical laboratory like the one advocated by Professor E. H. Moore of the Uni-

versity of Chicago. Teaching by experimental methods has already produced good results in promoting the metric system, and many cases are reported wherein school children are guiding their parents in adapting themselves to the new system. The construction of geometrical forms and graphic drawing are popular additions to teaching materials in all the schools.

II. The Secondary School

The Mathematical Conference of 1918. This conference played a very significant rôle in recent progress in the teaching of mathematics in the secondary school. Under the auspices of the Society for the Study of Secondary Education (organized for the purpose of investigating general subjects concerning secondary education) and through the efforts of Professor M. Kuniyeda, chairman of the committee of arrangements, and those assisting him, this conference was held in Tokyo for five days beginning on December 20, 1918. It was presided over by Mr. J. Kano, president of the Society, and was attended by about two hundred fifty mathematics teachers from all parts of Japan. At this conference, enthusiastic discussions and study were given on numerous subjects which were then at issue in connection with mathematical education in secondary schools, and proposals were prepared in response to the questions submitted by the Department of Education pertaining to improvements in the teaching of mathematics in compliance with the aims of education of normal schools, middle schools, and girls' high schools. Furthermore, important deliberations were made at this conference as hereunder enumerated, and resolutions were passed accordingly:

1. What are the proper measures to take in promoting mathematical ideas among the people?
2. What are the appropriate grades and times for teaching subjects relating to the function and to the graph?
3. What are the proper considerations for adding an introductory course in geometrical teaching and for using practical methods even on other occasions?
4. What is the particular consideration needed in making a close affiliation of the branches of mathematics in teaching?
5. What is the proper arrangement of each branch of mathematics and the allotment of teaching hours thereof?
6. What will be the necessary equipment for teaching mathematics?
7. Is it necessary to promote much wider use of soroban-calculation? [This question was answered affirmatively.]

The Mathematical Association of Japan for Secondary Education. At the conference it was proposed, at the instance of Professors T. Hayashi, M. Kuniyeda, the late Professors I. Mori, K. Hakii, and M. Kaba, to found an association for the study of mathematics and mathematics teaching in secondary education. This proposal was unanimously endorsed by those present and the association has since come into existence under the title given above. At the time of its foundation its officers were Professor T. Hayashi, president, Professors M. Mimori, M. Kuniyeda, vice-presidents, and the late Professor M. Kaba, chief editor. This association publishes an official organ, Vol. I, No. 1 having been issued in April, 1919. At present it has a membership of more than 2,300, and is very promising in its success and influence.

By making references to the resolutions passed by the conference in 1918, the association prepared teaching syllabi of mathematics for such secondary schools as middle schools, girls' high schools, girls' real-high schools, technical schools, commercial schools, and normal schools, and is constantly working to further the advancement of mathematics teaching in secondary schools by organizing it in effective ways.

Middle Schools. The middle school education aims at affording higher common education to boys who have completed the course of the ordinary elementary school. The course covers five years. Its graduates may immediately go out into the world, while some of them may enter various professional schools of higher grade or higher middle school to prepare for the university. (Completion of the fourth year qualifies one for admission to the higher middle school.)

According to the regulation, the chief aim of mathematics teaching in the middle school is to give the pupils necessary knowledge of mathematical quantities, to make them very skillful in computations, to render them efficient in applications as well as to train them for accurate thinking. The 1911 syllabus gives the teaching hours and arrangement of subjects as follows:

Subject	Year I	Year II	Year III	Year IV	Year V
Arithmetic	4	..	..	..	..
Algebra	..	4	5 (Algebra and Geometry)	4 (Algebra and Geometry)	4 (Algebra, Geometry and Trigonometry)
Geometry	..	..			
Trigonometry	..	..	..	..	

Of late a general tendency has been noted toward teaching algebra from the first year and geometry with its preparatory course from the second year. A new syllabus of mathematics in the middle school more adequately framed to conform with the demands of the times has been under contemplation by the Department of Education. At present a draft for remodeling the educational system of the middle school is pending with much discussion and it is expected that some significant changes will be made on the teaching syllabus which covers mathematics when the ultimate decision is made on the draft. The following is an extract from the detailed syllabus compiled in January, 1928, by the middle school attached to the Tokyo Higher Normal School and may serve to indicate the recent tendency in actual teaching.

EXTRACT FROM THE DETAILED SYLLABUS OF MATHEMATICS OF THE MIDDLE SCHOOL ATTACHED TO THE TOKYO HIGHER NORMAL SCHOOL

Year	Term	Arithmetic and Algebra	Geometry	Trigonometry
I	1	(4 hours per week) Integers, decimal fractions Weights and measures Four rules Area, volume Ratio, percentage, interest Miscellaneous problems in four rules Fractions Multiple, measure Four rules of fractions		
	2	Complex fractions Direct proportion, inverse proportion Miscellaneous problems in fractions Introduction to algebra Graphs		
	3	Four rules of positive and negative integers Addition and subtraction of integral expressions Coördinates Graphs of algebraical expressions		

EXTRACT FROM THE DETAILED SYLLABUS OF MATHEMATICS OF THE MIDDLE SCHOOL ATTACHED TO THE TOKYO HIGHER NORMAL SCHOOL—*Continued*

Year	Term	Arithmetic and Algebra	Geometry	Trigonometry
II	1	(2 hours per week) Linear equations with one unknown quantity Simultaneous linear equations with two unknown quantities Applied problems	(2 hours per week) Preparatory course	
	2	Multiplication and division of integral expressions Factoring	Triangle Polygon Parallel lines	
		Reduction to lowest terms Fractional equations		
	3	Reduction to the common denominator Four rules of fractional expressions Equations with literal coefficients Ratio, proportion Applied problems	Parallelogram Perpendicular, oblique line Circular arc, angle at the center	
III	1	(3 hours per week) Square root Irrational numbers Quadratic equations with one unknown quantity Equations of higher degree Irrational equations	(2 hours per week) Secant, tangent Segment Inscribed circle, circumscribed circle Two circles	
	2	Exponents and logarithms Compound interest	Loci Construction problems Algebraic expressions and geometrical construction Area	
	3	Variation of functions and their graphs (quadratic functions, quadratic equations) Direct variation Maxima and minima Inequalities	Area (continued) Ratio, proportion Proportional lines	

EXTRACT FROM THE DETAILED SYLLABUS OF MATHEMATICS OF THE MIDDLE SCHOOL ATTACHED TO THE TOKYO HIGHER NORMAL SCHOOL—*Continued*

Year	Term	Arithmetic and Algebra	Geometry	Trigonometry
IV	1	(2 hours per week) Variation of functions and their graphs (continued) Inverse variation Simultaneous equations of the second degree	(2 hours per week) Similar polygons Ratio of areas	(1 hour per week) Introduction to trigonometry Circular functions of acute angles
	2	Simultaneous equations of the second degree (continued) Arithmetical progression	Regular polygons and circles Circumference and π Fundamental properties of straight lines and planes Parallel planes and straight lines Perpendicular planes and straight lines	Circular functions of acute angles (continued) Circular functions of obtuse angles Properties of triangles
	3	Geometrical progression Annuities	Dihedral angle Principles of projective and perspective drawings Polyhedron, prism, pyramid	Solution of triangles Logarithms of circular functions Simple surveying
V	1	(2 hours per week) Permutation and combination Binomial theorem Elementary concepts of probability Functions and graphs including elementary concepts of the calculus	(2 hours per week) Volumes of prism and pyramid Regular polyhedron Circular cylinder, circular cone Volumes of circular cylinder and cone	(1 hour per week) Circular functions of any angle
	2	Reviews and supplements	Sphere Reviews and supplements	Circular functions of compound angles Solution of triangles
	3	Reviews and supplements	Reviews and supplements	Reviews and supplements

Notes:

(1) Each school year of this middle school is divided into three semesters, viz., the first from April 9 to July 10, the second from September 1 to December 24, and the third from January 8 to March 25.

(2) As the table shows, the teaching schedule of this school allots just one more hour for the fourth and the fifth year than is provided by the Department of Education.

The Present Phases of Mathematics Teaching in Middle Schools. Although the study of theory in teaching mathematics is a matter of much importance, it has become an essential aim to supply practical knowledge for the purpose of cultivating the pupils' ability to think mathematically. In selecting materials, specific precautions are taken to give materials suitable to the mental development of the pupils and applicable or interesting to them. Those which are too theoretical or abstruse for their understanding are to be avoided. In order to lighten the hardship which pupils encounter in learning the rudiments of algebra and geometry, preparatory courses are given in these subjects so as first to secure their interest in the new subjects.

In former days arithmetic, algebra, geometry, and trigonometry were taught quite independently of one another, but now efforts are being made to make close connection between these branches with the view of giving well-organized knowledge of mathematics to the pupils. Some discussions have been made on the disciplinary value of mathematics teaching; the opinions of American psychologists have been introduced, and this matter has attracted much attention on the part of our teachers.

The following points dealing with the more detailed phases of the course give further information as to the actual teaching at present:

1. Numerical computations are deemed very important; therefore drill is given in connection with numerical problems not only in the arithmetic of the earlier grades but also in the algebra and geometry of the higher grades. Stress is laid on approximate calculation, treatment of approximate values, computations by use of tables of logarithms and compound interest and also on the exercise of soroban-calculation. In addition, some teachers are inclined to teach the use of the slide rule.

2. Some materials which were formerly treated in arithmetic are now given with algebra as they can be more conveniently explained, and everyday computations which require much of worldly experience are given in the higher grades to more advantage, and consequently the arithmetical teaching in the lower classes is lessened, but it is generally believed not so advisable to exclude arithmetic from the lower classes in order to start early with the teaching of algebra and geometry.

3. Experiments, actual surveying, and other practical methods are used in teaching weights and measures, geometry, and other subjects. It is the general desire of schools to adopt the modern equipment which is necessary for mathematics teaching; and some schools are fitted with mathematical laboratories and special classrooms or good collections of models and specimens, though many are still insufficiently equipped.

4. Every teacher is very careful in teaching the functional relation between variable quantities and graphs, but there is still much to be desired in the study of the practical methods of their treatment.

5. Permutations, combinations, and the binomial theorem have been excluded from the syllabus of the Department of Education, but in some schools their outlines are taught, together with probability, since they are necessary for actual life problems. The opinion that it is necessary to teach the primary concepts of the calculus in the middle school has been supported by some but it is seldom tried.

6. As a result of adding new teaching materials to meet the requirements of the times, some of the old materials have been erased completely or are now treated very briefly. At the tenth annual meeting of the Mathematical Association held this year, it was proposed to exclude imaginary numbers from mathematics in the secondary schools. The idea was seconded by many but the final decision was against such a proposal.

7. The greatest obstacle to the complete realization of ideal improvements of mathematical teaching is the entrance examinations to higher institutions. The fact that such examinations often involve difficult questions with utter disregard of what is taught in secondary schools has naturally compelled the teachers in the middle school to cover hard problems of algebra and geometry for preparation for these entrance examinations. Such a condition is, of course, far from the original aims of secondary education. The Department of Education therefore gave out a general instruction requesting that the matriculation of higher middle schools be decided by taking into consideration the merits made in the middle school by the candidates and that the entrance examinations be made much easier.

8. Although in former days the lecture method was employed in teaching mathematics in the secondary schools regardless of whether the pupils really understood or not, many teachers are now using the heuristic method in teaching. Progress is being made with the individual mode of instruction by considering the individual needs and the development of the ability of each pupil, and in some schools the teaching is done by resorting partly to the laboratory method.

The present condition of the mathematical teaching in the other secondary schools is almost the same as that given above for the middle school.

Girls' High School. The girls' high school aims at providing higher general education for girls who have completed the ordinary elementary school. The course covers four or five years. According to the syllabus issued in 1911 by the Department of Education, each year of the course must have two hours a week for mathematics, mostly arithmetic, but in the upper classes simple algebraic expressions, equations, simple plane and solid figures may be taught in addition. It is maintained by many that in order to arouse enough pupil interest in mathematics and at the same time meet the demand of the times, more time must be given to the teaching of

algebra and geometry and that three hours must accordingly be allotted to each year for mathematics.

In 1920 the Department of Education revised the teaching schedule of mathematics of girls' high schools as given below, but no arrangement of the subject matter for this schedule has as yet been given out.

Course	Year I	Year II	Year III	Year IV	Year V
5-year	2	2	3	3	3
4-year	2	2	3	3	3

The tenth annual meeting of the Mathematical Association held last year passed a resolution as follows for recommendation to the Department of Education:

It is believed proper to allot three hours a week each year for mathematics, disregarding the length of the courses of the girls' high schools; and the standard of schedule of each branch of mathematics should be as hereunder given:

Subject	Year I	Year II	Year III	Year IV	Year V
Arithmetic	3	1			
Algebra		2	2	1	
Geometry			1	2	2
Reviews					1

(The schedule hours for geometry in the fifth year are to cover solid geometry and trigonometry also.)

It may be well to quote here a part of the resolution passed by the third annual meeting of the Mathematical Association held in 1921, which refers to the details of teaching mathematics in girls' high schools for the five-year course:

First year—three hours weekly—total 120 hours.

Arithmetic (120 hours): integers and decimal fractions; compound numbers (weights and measures, money, time, angles); properties of integers; fractions; ratio and proportion (simple proportion, compound proportion).

Second year—three hours weekly—total 120 hours.

Arithmetic (30 hours): ratio and proportion continued (proportional parts;* alligation); percentage; interest.

Algebra (90 hours): introduction; positive and negative numbers; four rules of integral expressions; linear equations with one unknown quantity; linear simultaneous equations.

* Subjects marked with * are to be treated briefly.

Third year—three hours weekly—total 120 hours.

Algebra (70 hours): factors; greatest common measure; least common multiple; fractional expressions; fractional equations; square root; quadratic equations with one unknown quantity; fractional equations.

Geometry (50 hours): preparatory course; triangle; polygon (side, angle, congruence, area).

Fourth year—three hours weekly—total 120 hours.

Algebra (60 hours): *simultaneous equations containing quadratic equations; ratio and proportion; arithmetical and geometrical progressions; miscellaneous.

Geometry (60 hours): circle (arc, chord, angle at the center, inscribed angle, segment, secant and tangent, two circles, inscribed and circumscribed figures, circumference and area of circle); proportion (proportional lines).

Fifth year—three hours weekly—total 120 hours.

Geometry (70 hours): proportion continued (similar figures, ratio of areas, circular functions of acute angles); straight line and plane; prism and pyramid; circular cylinder and cone; sphere.

Arithmetic, algebra, and geometry (50 hours): everyday computations (insurance, taxes, securities, annuities, installments, daily purchases, charges and fees, domestic accounts); reviews and supplements of preceding year's course.

Notes:

1. Soroban-calculation is to be exercised in each year by a timely arrangement of hours.
2. The graph is to be given by the proper arrangement and discretion of teachers in charge.
3. Simple geometrical construction and loci are to be given on all proper occasions.
4. In teaching arithmetic and geometry, efforts must be made to let pupils do experimental work and actual measuring.
5. Pupils must be made familiar with the use of letters while arithmetic or introductory lessons of algebra are taught.

The syllabus for the teaching of mathematics prepared by the same Association for girls' high schools of four-year courses is similar to that referred to above except that some of the subject matter of algebra and circular functions is reduced or omitted. In 1923 the Association prepared the syllabi on the teaching of mathematics for the postgraduate course of girls' high schools and the girls' real-high schools, and they will be discussed here.

It may be said that mathematics education in the girls' schools has made remarkable progress during the last decade, but while we hear much complaint concerning the difficulty experienced in teaching mathematics to girls we cannot deny that there is a tendency to teach in girls' high schools having excellent pupils the same

* Subjects marked with * are to be treated briefly.

grade of mathematics that is taught in the middle schools. Above all, however, it is more desirable that girls be educated to qualify for what is required of them in real life, because even though the number of girls desiring to enter the postgraduate course of girls' high schools or other higher institutions is ever increasing, it is still relatively small when compared with the number of boys who are candidates for higher education.

Normal Schools. The normal school is the institution for training the teachers of the elementary school, the class organization and the syllabus of which were revised in 1925 by the Department of Education in order to meet the requirements of modern times. This institution divides itself into two divisions: the first division extends over five years and admits those who have completed the first- or the second-year course of the higher elementary school; the second division for men is for the graduates of the middle school and its course covers one year, while the second division for women admits the graduates of girls' high schools and the course extends from one to two years. A postgraduate course was recently attached to those courses in order to provide the graduates more advanced education on selected subjects for the purpose of training superior teachers for the elementary school.

The syllabus of mathematics for the normal school for men is given below:

FIRST DIVISION

First year—four hours weekly.

Arithmetic and algebra

Integers, decimal fractions, fractions; proportion, percentage; negative numbers; integral expressions (addition, subtraction, multiplication, division); linear equations (linear equations with one unknown quantity, simultaneous equations, variation of values of linear expressions and their graphs).

Geometry

Construction of simple plane figures; making of models of simple solid bodies; mensuration of length, angle, area, and volume; straight lines (angle, parallel lines); rectilinear figures (triangle, parallelogram).

Second year—four hours weekly.

Arithmetic and algebra

Fractional expressions (factors, greatest common measure, least common multiple, reduction to the lowest terms, reduction to the common denominator, addition, subtraction, multiplication, division, fractional equations); quadratic equations (square root, irrational numbers, equations with one unknown quantity, fractional equations, irrational equations, simultaneous equations, variation of values of quadratic expressions and their graphs).

Geometry

Circle (arc, chord, angle at the center, inscribed angle, segment, tangent, two circles, inscribed and circumscribed figures); loci; construction problems.

Third year—four hours weekly.

Arithmetic and algebra

Proportion (ratio, proportion, direct variation, inverse variation, graphical illustration); progression (arithmetical progression, geometical progression); logarithms; percentage (percentage, interest, annuities).

Geometry

Area; proportion (proportional lines, similar figures); circumference and area of circle; circular functions, solution of triangles; simple surveying.

Fourth year—three hours weekly.

Algebra

Permutations and combinations; binomial theorem; probability.

Geometry

Plane and straight line (planes and straight lines, two planes, dihedral angle, solid angle); polyhedrons (prism, pyramid); round bodies (circular cylinder, circular cone, sphere).

Arithmetic

Various solutions of applied problems; method of constructing of problems; everyday computations; practice of soroban-calculation.

Method of teaching arithmetic in elementary schools

Aim; selection and arrangement of subject matter; methods of instruction; instruments and useful advices for actual teaching; study of textbooks of elementary schools.

Fifth year—three hours weekly.

Reviews and supplements

Relation between number and quantity; numbers, algebraical expressions; equations, inequalities; functions and their graphs; maxima and minima; general method of finding the greatest common measure; evolution (square root, cube root); principles of projective and perspective drawings; conic sections; ellipsoid.

SECOND DIVISION

First year—two hours weekly.

Arithmetic

Various solutions of applied problems; method of constructing problems; everyday computations; practice of soroban-calculation.

Method of teaching arithmetic in elementary schools

Similar to those taught in the first division.

Postgraduate course—four hours weekly.

Algebra and geometry

Permutation; combination; probability; progressions; logarithms; theory of equations; loci; problems in geometrical construction.

Trigonometry

Circular functions, trigonometrical equations, spherical triangle.

The outline of analytical geometry
The outline of calculus
Study of the teaching arithmetic in elementary schools

Note: Among the remarks attached to this syllabus it is stated that enough exercise in the computation of numbers should be given in every year course and that pupils must become well versed in approximate calculation and treatment of approximate values and familiar with the use of the slide-rule and tables of logarithms and compound interest, and that teaching of loci, construction, and area should be given also in other school years than those provided in the syllabus by proper arrangement.

The teaching subjects for the first division of the normal school for women in the revised syllabus are the same as those of the normal school for men except that the teaching hours of mathematics in the third year are one hour less.

Under the old system, the teachers and the pupils in the normal schools were considerably overburdened because general knowledge of mathematics had to be treated within a comparatively few hours. Moreover, the study of the teaching of arithmetic in the elementary school had to be taken as well, but the revised syllabus increased the teaching hours and made the instruction far more substantial. Such amendment of the syllabus was, of course, made by comparison with the proposed syllabus prepared in 1924 by the Mathematical Association. Since no detailed subject matter for the post-graduate course was contained in the syllabus, the details have been a subject of discussion and study of some members of the Association at every annual meeting for the last few years.

III. Higher Middle School

Course of Study. The whole course of the higher middle school extends over seven years. The first four years are for the ordinary course, the schedule of which is same as that of the first four years of the middle school; and the higher course covers the remaining three years. Although some higher middle schools of seven-year courses have been established in recent years, most of the higher middle schools have the higher course only.

The higher course is subdivided into a literary course and a science course. In the first year of the literary course three hours per week are allotted to mathematics, and according to the syllabus of 1923 of the Department of Education supplements of geometry (solid) and algebra and the outlines of trigonometry, plane analytical geometry, and calculus are to be taught.

In the science course four hours per week are allotted to mathematics in each year, and the syllabus of 1926 of the Department of Education provides solid geometry (about 20 hours), trigonometry (about 40 hours), analytical geometry (plane and solid, about 70 hours), algebra (about 60 hours), and calculus (about 170 hours); in addition, dynamics (2 hours per week) can be taught to those who may desire it.

I have given a summary of the main changes recently made in the teaching of mathematics in the various schools for ordinary education. In conclusion I wish to express my warm thanks to my colleagues, Professors T. Ando, R. Kurokawa, and others for their generous help in preparing this report.

RUSSIA

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The Reform Project of 1915. Materials relating to the teaching of mathematics in Russia were assembled by the Russian delegates to the IMUK Conference and were published in 1909-1912. Since that date, the teaching of mathematics has seen far-reaching changes, of which I shall attempt to give an account.

In the first place, mention should be made of the chief innovation made by the Russian delegation in the period under discussion. This was the addition of a special course in the seventh class of the *Realschule*,[1] consisting of the ideas of analytic geometry and the infinitesimal calculus. Syllabi of this work are given in the memoir by K. Vogt [2] and also in my report in *L'Enseignement mathématique*.[3] Teachers were divided in their opinions as to the result of these innovations. The conservatists found that higher mathematics was too difficult for the students of the secondary schools. The advocates of the reform, on the other hand, considered parts of the course deplorably simple and urged that they be included in elementary mathematics. Some of the professors in the higher technical schools saw an appreciable gain from this innovation; others, on the contrary, felt that the first-year students had lost the interest which always attaches to a totally new subject without obtaining the solid foundation necessary to their later studies. Sometimes, also, there was the additional difficulty of having to learn again things which might or might not have been taught in the secondary schools.[4] Be this as it may, the innovation pro-

[1] The *Realschule* corresponds to the *Gymnasium* in Germany with the difference that modern languages replaced the classics. Its course followed the first three years of the primary school and its seven or sometimes eight classes included pupils from the age of ten to eighteen. The course in the *Realschule* was paralleled by the eight years of the modern or of the classical *Gymnasium*.—Editor.

[2] *Bericht über d. mathematischen Unterricht an d. russischen Realschulen.*

[3] Vol. XI (1909), pp. 33-35.

[4] M. J. Popruginko notes in his report on the teaching of mathematics that the teaching of analytic geometry in the cadet corps had had no noticeable results (*resultats positifs*).

duced a leavening in the stagnant form of the teaching of mathematics and it was by no chance coincidence that the calling of a congress of teachers of mathematics was proposed at the same time to consider the teaching of the subject. This conference took place in St. Petersburg,[5] January 9-16, 1912. Two years later, this was followed by a second conference at Moscow, January 8-16, 1914.[6] During the interim, the Thirteenth Congress of Russian Scientists and Physicians took place at Tiflis. This differed from their preceding meetings by the addition of a very active section on the teaching of mathematics. The lively interest in the subject resulted in the founding of a journal which appeared at Moscow under the editorship of J. J. Tchistiakov.

It is not without significance that the matter of the reform of the secondary schools was mentioned in the general orders given by the Minister of Public Instruction, and that in April, 1915, a special conference on this subject convened under the presidency of M. N. P. Ignatieff. The general plan adopted by this conference was as follows:

1. Schools to be state-supported.
2. A liberal education to be given, not one having preparation for the higher schools as its immediate objective.
3. The school course to be of seven years, divided into two cycles of three and of four years each, the teaching in the first cycle to conform to that of the existing higher primary schools so that there would be but one syllabus for this cycle; specialization to begin in the second cycle.

In each year of the first cycle, four hours a week were assigned to mathematics, making a total of twelve hours. Four courses were planned for the second cycle in imitation of the French system. These with their time-allotment in mathematics were as follows:

1. Neo-humanitarian, one modern foreign language, emphasis on literary studies, mathematics 4, 4, 4, and 3 hours a week respectively in the four years. Total 15 hours.
2. Classical-humanitarian, one modern foreign language and one ancient language, preference given to literary subjects rather than to mathematics and science. Mathematics 4, 3, 3, and 2 hours. Total 12 hours.
3. *Realschulė*, specializing in natural science, mathematics 4, 4, 4, and 5 hours. Total 17.
4. *Realschule*, specializing in mathematical sciences, mathematics 4, 6, 6, and 6 hours. Total 22 hours.

[5] See my report of this meeting in *L'Enseignement mathématique*, XIV (1912), pp. 222-28.

[6] See *Zeitschrift für mathématischen und naturwissenschaftlichen Unterricht*, XLV (1914), pp. 300-304.

Thus, the total time assigned to mathematics for the entire course was (1) 27 hours, (2) 24, (3) 29, and (4) 34. In comparison with the distribution of hours in the *Gymnasium* as it then existed, where 32 hours were allotted to mathematics in the eight classes, or 30 in those schools which had no eighth class (a class which was given over principally to review), and in contrast to the *Realschule* with 35 hours devoted to mathematics,[7] it is evident that the reform proposed in 1915 favored the classics rather than mathematics, reducing the latter almost to the standard of the six-year commercial courses in the *Realschule* in the case of the neo-humanitarian course, and in the scientific course diminishing the time until it was comparable to that of the former classical *Gymnasium* (29 as compared with 32). It was only in the mathematics section of the *Realschule* that the proposed plan preserved practically the same number of hours as that in the former schools (34 as against 35).

The materials collected by the commission were published by the Ministry (Department) of Education in 1915.[8] These included syllabi in mathematics with explanatory notes written by a specially appointed subcommittee. In this report there appeared an expression of opinion from K. Lebedintseff, who thought it might be possible to introduce the elements of higher mathematics into the neo-humanitarian course and who drafted a program of studies on this plan including the theory of combinations, the binomial theorem, continued fractions, and indeterminate equations. There also was a memoir from Professor D. A. Grave, who protested against the further lessening of the mathematics taught in the classical course which included two ancient languages.[9] Another of the contributors was W. W. Kondratieff, director of the eighth *Gymnasium* at St. Petersburg, who considered it indispensable to teach the seventh class the theoretical foundations (*theoretische Begründung*) of the four operations with whole numbers and fractions. Framed on a schedule of hours that the subcommittee on mathematics found insufficient, these syllabi were in the nature of compromises and they met with lively criticism at the hands of

[7] Commercial courses extended only through the sixth year and the mathematics of the fifth and sixth years was reduced to 2 and 3 hours respectively, leaving a total of 23 hours.

[8] *Materialy po reforme sredney shkoly. Primernye programmy y obyasnenia.*

[9] A number of such schools were proposed which were not included in the four divisions listed above.—Editor.

the methodologists at Moscow especially.[10] These innovations were scarcely put into practice, however; for a year later the revolution took place, everything was overturned, and the system of public education was radically changed. It is hardly worth while to consider the work of the Ignatieff Commission at greater length. I have described it only because it shows that even at that time there was a tendency to minimize the place given to mathematics in the curriculum—a tendency that has characterized the post-revolutionary period, although naturally people have given it a different theoretical explanation.

In the U.S.S.R.,[11] there exist two principal systems of public instruction: that of the R.S.F.S.R.[12] (Russia) and that of the Ukraine. The other republics of the union have imitated one or the other of these, generally choosing that of the R.S.F.S.R. Accordingly, I shall describe these two systems only.

Public Education and the Teaching of Mathematics in the R.S.F.S.R. The history of public education in Russia after the revolution of 1917 falls into three periods:

1. The period of the provisional government when public education had not been affected by radical reforms.
2. The period following the Communist party's coming into power, the period of the communistic government or of the "SEP," [13] the period of the destruction of the old institutions and of the eager and feverish construction of the new. Universities and technical colleges were thrown open to the entire proletariat. Each town founded a university of its own. It was not long, however, before the greater part of these ephemeral creations died a natural death, but even so the number of universities and higher technical schools was considerably increased, providing a sufficient number for the third period referred to below.
3. The period of the consolidation and construction, called the "NEP." [14]

[10] See the Russian journal, *The Teaching of Mathematics*, for 1915.

[11] Union of Socialist Soviet Republics, the new legal name for "Russia," consisting of six republics: the R.S.F.S.R. (Russia proper), White Russia, Ukraine, Transcaucasus Federation, Usuzbek, and Turkoman.—Editor.

[12] Russian Socialist Federated Soviet Republic.—Editor.

[13] The old economic policy, these words in old Russian being *"Staray Economitcheskaya Politika."*

[14] The new economic policy, *"Novaya Economitcheskaya Politika."*

It should be noted that certain of the schools had moved because of the evacuation of institutions in regions held by the enemy. Thus, the University of Jurieff was transferred to Voronej, and that of Varsovie to Rostov (on the Don). Since a certain preliminary preparation seems indispensable for students admitted to the higher schools, this emergency called into being a preparatory course for the higher institutions. This was at first called the "O-semester"; later it was called the class for workers (*facultés ouvrières*). It seemed probable that these classes would be temporary, lasting only until the reorganized secondary schools would put a limit on the number of new students. At the present time, however, no reduction in their numbers is perceptible. On the contrary, they show signs of further development and their two-year course is being extended to three years.

The primary and secondary schools have undergone radical reorganization. In the R.S.F.S.R. a unified activity school [15] has been organized. This has a nine-year course, the first four years of which form the first division or cycle. In the first cycle, the curriculum is built on a scheme of complexes,[16] that is, the teaching of a class in the hands of a single master. The subject matter to be taught, however, is not divided according to the traditional rubrics of language, natural and physical sciences, mathematics, etc., but is grouped according to certain topics.

The First Cycle. It is exceedingly difficult to state the time given to the teaching of mathematics in the primary schools (that is, in the first four years) because of the complexes about which the work centers, and one can only give a few conclusions based on the general introduction that appears in the syllabus [17] and then give the program in mathematics.

What place, then, is assigned to the teaching of reading, writing, and arithmetic? Providing the child with the knowledge of reading, writing, and computation is one of the most important of the school's problems. If this knowledge is not provided, it is impossible

[15] This school is also called a "unified labor school" from the *einheitliche Arbeitschule* which is the name given it in discussions written in German. This term, however, is misleading to American readers as it connotes socialism rather than a method in education.—Editor.

[16] The term *complex* is somewhat equivalent to the "unit of work" idea now gaining popularity in elementary schools in America. It is the center of interest about which are clustered all of the activities of a class.—Editor.

[17] *Programs and Methods of the Unified Activity Schools,* Vol. I, 1928. (I shall give a brief exposition of the parts which bear on mathematics.)

to speak seriously of the socialistic education of the mass of the children. The vital aspect of the work of the schools, that is to say, the carrying out of its unified program which consists both in raising the level of the development of the child and in increasing his value to the country, all enhances the importance of the knowledge of reading, writing, and computation. The main reasons for the lowering of the standards of instruction are the poverty of the school and a badly organized school year. But in addition there is the dearth of information on method on the part of the teachers; many of them know the traditions of the former schools but have only a superficial acquaintance with the newer methods. The new syllabi in languages and in mathematics give precise specifications on the extent of knowledge and the corresponding techniques to be acquired in the different years of instruction. The explanations of the methods to be used aid in the understanding of the intimate connection between work in languages and mathematics and the general work of the complexes and the general life of the pupil community. It is impossible to indicate the percentage of time which should be devoted to acquiring the essential knowledge of language and mathematics outside of the complexes, for it varies from one school to another.

SYLLABUS IN MATHEMATICS

First Year

Numbers belonging to the first decade. Counting forwards and backwards. Addition and subtraction of numbers less than 10.

Numbers of the second decade. Counting. Four rules with numbers less than 20 (i.e., addition, subtraction, multiplication, and division).

Concept of parts of a unit, half, fourth. Writing of ½ and ¼.

Counting by tens. Numbers to 100.

Meter and centimeter, kilogram, litre. Uses of these measures.

Telling the hours on a clock, days of the week. Counting of specie, paper money.

Geometric concepts: square, triangle, circle. Drawing these figures.

Position of an object with respect to another: left, right, above, below, etc.

Solving of problems of one or two questions each.

Second Year

Counting backwards and forwards to 100. Addition and subtraction of numbers within the limit of 100.

Multiplication and division by tables.

Comparison of numbers by difference and quotient, greater, less, too much, too little, how much and how many times.

Counting by hundreds. Four operations with 100, 200, etc. Numbers to 1,000.

Addition and subtraction of numbers in this domain. Unit fractions, ½, ¼, ⅛, $\frac{1}{10}$. Transformation of fractions. Addition and subtraction of fractions.

Measures of length (meter), weight (gram), time (day, month, year).

Staking out straight lines on the ground and measuring them.

Making change.

Addition and subtraction of denominate numbers of two denominations, as ruble and copek, meter and centimeter.

Telling time to minutes. To tell the time elapsed between two events within a month.

Equality of sides and of angles in a square. Units of measure of areas (square meter, decimeter, and centimeter). Measuring the area of a rectangle, as for instance the classroom.

Problems containing two or three questions.

Third Year

Four operations with numbers less than 1000. Numbers to 1,000,000. Separating numbers into orders and classes. Addition and subtraction of numbers to 1,000,000. Division of numbers as far as 1,000,000 by numbers with 1, 2, or 3 decimal places. Division by a number of several figures when the quotient is a one-figure number. Use of s'choty[18] (*tables à calcule* [*russe*]) for the addition and subtraction of numbers of several figures. Multiplication and division of compound and denominate numbers.

Beginning of decimal fractions. Oral and written naming of decimals. Transformation and comparison of decimals. Addition and subtraction of decimals not smaller than hundredths. Writing compound denominate numbers as decimal fractions.

Finding the time elapsing between two events within a year of each other.

Equality of sides and angles of a rectangle. Drawing parallels and perpendiculars on paper or laying them out on the ground. Measuring the area of a rectangular field. Units of land measure (are, hectare, arpent). Making a map of a field. Drawing straight line diagrams (not more than three or four). Idea of scale drawing. Equality of faces of a cube.

Measures of volume (cubic meter, decimeter, centimeter). Measuring volume of rectangular box.

Problems containing three or four questions.

Fourth Year

Decimal fraction: multiplication and division of a fraction by a whole number, of a whole number by a fraction, of a fraction by a fraction. Multiplication and division by 10, 100, 1,000.

Idea of per cent. Simple interest.

Common fractions: changing one fraction into another. Comparison of fractions, reduction of fractions. Addition and subtraction of fractions by inspection (denominators less than 100). Finding part of a given number or finding a number, knowing a given part. Changing common fractions to decimals. Approximation to within 0.1, 0.01, 0.001.

Arithmetic mean. Knowledge of the terminology of arithmetic operations and of the relation between terms.

[18] The Russian abacus.

Changing Russian measures into metric units. Use of conversion tables.

Determination of the interval between two dates within a century.

Keeping accounts of receipts and expenditures of money and of goods.

Drawing circles. Relation of radius, diameter, and circumference. Units of angle measure. Measurement of angles. Drawing angles with a protractor. Simple diagrams with a circle divided into 2, 3, 4, 6, 8 parts.

Properties of sides and angles of rectangles, isosceles triangles, equilateral triangles. Area of a triangle. Making a simple contour map. Use of scales in measuring distances between points on a map or a plan.

Right prism, volume of a prism, cubical contents of a room, of a house.

This syllabus is followed by a discussion of the connection of mathematical skill in practical affairs which we will abridge as follows:

To show the children how to use computation and measurement to interpret the life about us, it is necessary to build habits of accurate calculation and measurement, to stimulate their ideas and to add to their economic value by the ability to keep expense accounts and to compute discount. Skill in computation is acquired by the solution of a sequence of problems (with numbers in the first decade to begin with, then numbers from the second, etc.), these problems dealing with material that is varied systematically. The method of analogies is used to a great extent. The study of operations with numbers in the thousands begins with simple examples. The recognition of the commutative properties of a product and of a sum are helpful in this work. Each type of computation exercise is carried out in three ways: mental problems, columns or parentheses, written problems.[19] The content of the mental and verbal problems is drawn from the units of work given in the school, although the actual experience of the pupils may not include this work. Too much time should not be given to drill exercises. Several years ago the work with compound denominate numbers was greatly abridged for this reason. At present it has been proposed that operations with numbers of several figures and with fractions may be lessened without fear of harm either to the child's mental development or to his usefulness. In the first cycle, it is generally sufficient that the pupils can multiply two three-figure numbers or divide by a three-figure number. With the introduction of the metric system, the schools should give more attention to decimal fractions. As for common fractions, it is sufficient that the students acquire

[19] We should perhaps say verbal problems solved mentally, abstract practice exercises, and verbal problems solved with a pencil.—Editor.

skill in adding and subtracting fractions by inspection without the use of the least common multiple.

The solving of a given problem affords an opportunity for acquiring knowledge of making an analysis. Life demands another skill —that of discovering the given data and of stating the relations between them. Accordingly, the activities are organized as seems suitable. For the first group, the activities are not so long as a half year. In the second group, the accounts of the fodder for a cow are kept; in the third, the accounts of a harvest from which several problems may be deduced: to calculate the period of vegetation, the harvest per arpent, and like problems. It is necessary to emphasize the relation of these results to mathematics itself (as in the properties of operations). In the third year, the examples are written as exercises in arithmetic; in the fourth, the pupils are already constructing practical formulas. The elements of geometry are taught in connection with arithmetic. The course also includes the recognition of geometric forms and the study of solids. The properties of these figures are studied by measurement, as, for example, finding the sum of the angles of a triangle by using a protractor. With the older children the concepts of geometry are developed as a basis for work in that subject. It is absolutely necessary to learn the making of the simplest surveys, to know how to use the pace scale in measuring distances, to have practice in gauging distances by the eye. During the first three grades, lengths are measured by the pace. In the fourth year, simple surveying instruments are used. Geometry should be connected with arithmetic. Graphic methods are highly commended. One should be somewhat conservative in the making of graphs. Curved line graphs are made only in the case of temperatures. Straight line graphs are to be used, this work being enlivened by the making of geometric designs. The drawing should be neat and exact but the number of drawings should not be large. Arithmetic problems may be solved by this method, but it should not be greatly stressed, in view of the fact that the development of skill in computation is one of the major problems of the school. At the end of the report is a discussion of models and instruments, the abacus, multiplication table, slide rule (*regle à calcul*). Finally, the syllabus is analyzed by years, giving the topics which may be omitted.

The Second Cycle. After four years' study in the primary school, the pupil enters the secondary school, which is divided

into two parts. The first division of three years (fifth, sixth, and seventh) forms the second part of the seven-year schools found both in the towns and in the country. In reality, however, the country schools consist of but the first three classes (first to third). The eighth and ninth years, which are the second subdivision of the secondary schools, are expected to give an average education to the majority (75 per cent) of the pupils, only 25 per cent of the students going to the higher schools. (This is according to data from the Commissariat of Public Instruction.) Thus, in the plan of study and in the organization of the curriculum the predominant problem is the continuation of the general education which the seven-year school gives. The second objective is to complete within the period of nine years of academic work a general education and to lay a foundation for a materialistic concept of the world. "Since the first objective depends on the second, the second end of school is thus made clear." At the same time the sum total of the knowledge gained by the pupil in the second should be entirely sufficient for the continuance of his studies in one or another of the advanced schools.

In regard to the purpose of the second division, as is the case in the first, there is a change in the idea of a general education, regardless of the precedent of the pre-revolutionary schools, and a shift of the emphasis to the social sciences on the one side and to the natural sciences in the larger sense of the word on the other. It would be more correct to say that this tendency is apparent in such studies as physics, chemistry, natural science, and mathematics, which occupy less than a third of the student's weekly program.

The requirements of the higher schools, however, have provoked certain tendencies toward lessening the number of hours given to chemistry and to the natural sciences in favor of mathematics, but it would be unjust to consider this tendency as an outgrowth of the desire to complete an average education in a nine-year course.

The data in the first table below concerning the program of studies of a seven-year town school are from the official record of a mathematics teacher in a school which has not attained the level necessary for the pupils to enter the higher school. The reader can get some idea of the nature of the work done in this type of school by making a more or less detailed study of the table.

PROGRAM OF STUDIES OF A SEVEN-YEAR TOWN SCHOOL

Course	5th Year	6th Year	7th Year	Total Number of Lessons *
1. Social Sciences	4	4	4	408
2. Language	5	5	4	476
3. Mathematics	4	4	5	442
4. Natural Sciences	3	4	4	374 †
5. Chemistry	1	2	2	170 †
6. Physics	4	4	4	408
7. Geography	2	2	2	204
8. Foreign Languages	3	3	3	306
9. Shopwork (*travail*)	3	3	3	306
10. Graphic Arts	2	2	2	204
11. Singing, Music, Rhythm	2	1½	1½	170
12. Physical Education	2	1½	1½	170

* The last column is calculated on the basis of 36 weeks a year, of which one is devoted to organization and one to final work.

† Prior to the 5th school year, these subjects are taught by the same instructor.

PROGRAM OF STUDIES OF A SECONDARY SCHOOL—SECOND CYCLE

A. STUDIES COMMON TO ALL COURSES

Course	8th Year	9th Year	Total Number of Lessons
1. Social Sciences	5	4	306
2. Language and Literature......	4	4	272
3. Mathematics	4	4	272
4. Natural Sciences	3	3	204
5. Chemistry	2	2	136
6. Physics	3	3	204
7. Modern Foreign Languages...	2	2	136
8. Graphic Arts	2	1	102
9. Singing, Music, Rhythm	2	1	102
10. Physical Education	2	1*	102
Total	29	25	1836

* Two hours in the course in pedagogy.

B. STUDIES BELONGING TO PARTICULAR COURSES

Course	No. of Hours in 8th Year	No. of Hours in 9th Year	Total No. of Hours
Pedagogy			
a. Academic	9	12	714
b. Pre-school	8	12	680
c. Political Instruction	9	12	714
Coöperative Course			
a. Rural (*Coöperativeagricole*)	9	13	748
b. Consumer	9	13	748
Administration Course			
a. Duties and Finance	7	13	680
b. Insurance	7	13	680
c. Credit	7	13	680

SYLLABUS IN MATHEMATICS FOR THE FIRST PART OF THE SECOND DIVISION OF THE SECONDARY SCHOOL

Fifth Year

Review and generalization of the pupil's knowledge of decimal notation, on arithmetic operations with integers of considerable size and with decimal fractions. Parts of a number expressed in per cents.

Relation between the given quantities and the results of operations. Checks for computation. Solution of simple equations by the relations thus established. Effect on results of changes in numbers themselves.

The straight line, segments, measurement. Metric measures of length. Use of ruler. Construction of straight line diagrams and graphs of empirical data. Estimating errors of measurement. Rounding off the results of operations on approximate numbers.

Common fractions. Reducing fractions to lowest terms. Prime and composite numbers. Conditions of divisibility by 2, 4, 5, 3, 9. Factoring. Least common multiple. Operations on common fractions. Changing fractions to decimals equivalent within an assigned degree of accuracy. Drill in computation with common fractions and decimals combined.

Right angle and its parts as a measure of angles. Circumference, measurement of arcs in degrees, measurement of angles by their arcs, use of the protractor. Construction of an angle equal to a given angle by ruler and compass. Addition and subtraction of angles. Multiplication and division of angles by a given number (by construction). Construction of designs using circles. Adjacent angles. Equal adjacent angles. Perpendiculars and obliques. Supplementary angles. Sum of adjacent angles. Sum of angles about a point. Vertical angles.

Rectangle, square, parallelogram, triangle, trapezoid. Principal properties of these figures. Their construction with ruler and square. Metric units of area. Area formulas for the figures mentioned above. Area of polygon by breaking it into triangles and trapezoids.

Use of letters to indicate operations on the numbers represented by the

letters. Idea of a power. Side of a square (of given area) given by trial and by tables (of square roots). Formulas for the solution of problems. Order of operations. Use of parentheses. Evaluation of formulas for integral and fractional values of the letters. Cube and rectangular prism. Metric units of volume. Volume formulas for cube and rectangular prism. Formula for computing the volume from the weight. Metric units of weight. Area and volume of a triangular prism.

Ratio. Ratio of segments, areas, volumes, weights, etc. Scales and the conversion of linear scales into number scales. Ratio expressed as per cent. Relative error expressed in per cent. Direct variation. Properties of mean proportions and the determination of the unknown term. Inverse variation. Division of a magnitude into parts having a given ratio. Problems on dividing a magnitude proportionally.

Solution of simple equations in one unknown having numerical coefficients.

Circle and cylinder. Experimental determination of the ratio of a circumference to its diameter. The number 3.14. Formula for the length of a circumference. Experimental determination of the area of a circle. Formula for the area of a circle. Study of a cylinder. Formulas for area and volume.

Construction of a triangle having three sides given. Knowledge of conventional notation.

Geodetic work:

(1) Survey of a field with simple contour by measuring lengths alone (by breaking the area into triangles).
(2) Survey by aid of the square.
(3) Mapping a field with simple contour or a footpath by use of a plane table, compass, and triangle.
(4) Making a profile with a triangle and a surveyor's rod, a spirit level or a mason's level.

Sixth Year

Axis of numbers. Graphic representation of quantities whose direction is opposite to each other. Signed numbers. Absolute value. Addition and subtraction of signed numbers. Multiplication and division of signed numbers Solution of equations by transposing terms.

Parallel lines. Construction with ruler and square. Equality of corresponding angles and of alternate interior angles. Theorem of angles whose sides are parallel. Properties of angles whose sides are perpendicular.

Power of a number. Finding integral or fractional powers. Powers of numbers expressed by letters. Rules for multiplication and division of positive powers of numbers having the same base. Commutative law of multiplication ($ab = ba$). Associative law $(ab)c = (ac)b = a(bc)$. Multiplication of monomials. Changing fractions having monomial numerators and denominators to equivalent fractions. Solution of first-degree equations with two terms and with literal coefficients. Addition and subtraction of monomials. Four rules for dealing with algebraic fractions with monomial numerators and denominators.

Position of a straight line with reference to a circle. Position of line with reference to two circles. The conditions under which a triangle is determined.

Construction of a triangle, having its three sides given. Construction of an angle equal to a given angle by means of a ruler and compass. Construction of a triangle from other elements. Conditions that govern the equality of triangles. Theorem of the sum of the angles of a triangle. Classification of triangles according to their angles. Isosceles triangle. Concept of symmetry. Theorem of the bisector of the angle of an isosceles triangle. Construction of axis of symmetry and of the bisector of an angle. Perpendiculars, obliques and their projections. Distance from a point to a line. Erecting and dropping perpendiculars. Division of a segment into two equal parts.

A polynomial as an algebraic sum. Commutative law of addition $a + b = b + a$. Associative law of addition $a + (b - c + d) = (a + b - c) + d$. Addition and subtraction of polynomials. Solution of equations. Distributive law for multiplication. Multiplication of a polynomial by a monomial, geometric illustrations. Division of a polynomial by a monomial. Taking out a common factor. Simplification of algebraic fractions with polynomial numerators and denominators by the removal of a monomial factor. Solution of equations of the first degree with literal coefficients. Multiplication of one polynomial by another. Geometric illustrations. Factoring a polynomial by grouping. Special products $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$. The use of special products in the multiplication of two numbers. Factoring by means of the formulas for special products. Computation of the difference of two squares by factoring. Simplification of algebraic formulas by combining fractions. Multiplication and division of fractions having polynomial numerators and denominators. Solution of fractional equations.

Quadrilaterals. Theorem for the sum of the angles. Axis of symmetry of isosceles trapezoid. Parallelograms. Theorems regarding sides, angles, diagonals. A rectangle as a special case of a parallelogram, axis of symmetry, theorem regarding diagonals of a parallelogram. Rhombus as a special case of a parallelogram, axes of symmetry, properties of diagonals. Square as a special case of rectangle or rhombus. Axes of symmetry. Construction of quadrilaterals of different types according to certain given conditions. Theorem regarding the line which bisects two sides of a triangle. Division of a segment into equal parts. Problems in computing linear elements and areas of quadrilaterals by means of equations. Graph of functions $y = ax$, $y = ax + b$, and of equations in the form $y = ax + b$. Transforming equations in the form $ax + by + c = 0$ into the form $y = ax + b$. Simultaneous linear equations of the first degree. Graphic solution of equations. Solutions by substitution and by comparison. Solution of three simultaneous linear equations in three unknowns. Solution of problems by means of equations.

Circle and circumference. Axial and central symmetry of a circle. Theorem concerning the diameter that is perpendicular to a chord. Relation between the length of a chord and its distance from the center of the circle. Properties of the radius to the point of contact of a tangent. Construction of a line tangent to a circle at a given point on the circumference. Measurement of angles and arcs. Central angles. Measure of angles inscribed in a circle. Angle inscribed in a semicircle. Construction of a line tangent to a circle and passing through a given external point. Angle made by a tangent and a chord passing through the point of contact. Problems dealing with the measurement of the

circumference and arcs of a circle, also with areas and sectors. Making of designs on different scales. Concept of similarity.

Geodetic work (optional):

(1) Measure of angles in a field with the simplest goniometer and the application of this work to mapping a field with a simple outline.
(2) Making a profile by using a simple level.
(3) Finding the height of an object by similar figures.

Seventh Year

Determination of the ratio between two line segments. Construction of similar triangles. Center of similitude. Properties of parallel lines which cut the sides of an angle. Construction of a fourth proportional. Construction of similar quadrilaterals. Conditions of similarity. Ratio of the areas of similar triangles and polygons. Concept of sine, cosine, and tangent of an acute angle. Variation of trigonometric functions with changes in the angle. Three-place tables of trigonometric functions. Solution of right and isosceles triangles by trigonometric functions. Formula for the area of a triangle in terms of two sides and the included angle.

Simple methods of calculation with rounded numbers.

Calculation of the side of a square having a given area. Square roots of whole numbers and decimals carrying the result to a unit of given order.

Pythagorean Theorem. Mean proportional. Relation between the height of a right triangle and the projections of its sides. Algebraic relations between the sides of a right triangle. Relation between the perpendicular dropped from a point on a circumference to a diameter and the projections of the chords on the same diameter. Relation between the lengths of a tangent, a secant, and the external part of the secant. Construction and calculation of a mean proportional.

Relations between the trigonometric functions of an acute angle:

$$\sin^2 x + \cos^2 x = 1 \qquad \tan x = \frac{\sin x}{\cos x}$$

Incomplete equations of the second degree and their solution. Taking factors from under the radical sign. Inserting factors beneath a radical sign. Extracting square roots of a fraction whose numerators and denominators are monomials or the perfect squares of binomials. Graphic study of the function $y = ax \pm b$. The meaning of the constants a and b. Graph of $y = \frac{a}{x}$ for different values of a. Concept of function $y = ax^2$ as an expression of the variation of a quantity proportional to the square of another quantity. Graph of this function for different values of a.

Construction and solution of equations of the second order: two roots of the equation, solution by completing the square. Formula for the solution.

Concept of inscribed and circumscribed polygons. Construction of circle circumscribed or inscribed in a triangle. Properties of the angles of an inscribed quadrilateral. Regular polygons. Construction of the following inscribed regular polygons: triangle, quadilateral, hexagon. Finding the value of the side of each in terms of the radius of the circumscribed circle.

Computation of trigonometric functions of angles of 30°, 45°, and 60°. Construction of regular octagon and dodecagon by doubling the sides of a square or of a regular hexagon inscribed in a circle. Axial and central symmetry of regular polygons. Size of their angles. Trigonometric formulas for the sides and area of a regular polygon in terms of the radius of the inscribed or circumscribed circles.

Calculation from formulas of the area and volume of the following solids: right prism, cylinder, cone, sphere. Application of these formulas for the area and volume to the solution of problems calling for the use of approximate computation. Idea of representing these solids by their orthogonal projections.

Geodetic work: Determination of inaccessible heights and distances by finding the angles with a goniometer and by computing the results graphically or by means of trigonometric functions.

PROGRAM OF STUDIES OF THE SECONDARY SCHOOL
SECOND CYCLE
(8th and 9th years)

Eighth Year

Summary of work with fundamental operations on algebraic polynomials. Proof of the identity

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b.^3$$

Derivation of formulas for

$$\frac{a^3 - b^3}{a - b} \quad \text{and} \quad \frac{a^3 + b^3}{a + b}.$$

Simple transformations and operations with expressions which contain the radical sign. Bringing numbers under the radical sign, reduction to like roots, multiplication, division of radicals, raising them to powers and extracting roots. The simplest cases of clearing the denominator of a fraction of radicals:

$$\frac{a}{\sqrt{b} \pm \sqrt{c}}, \quad \frac{a}{m\sqrt{b} \pm n\sqrt{c}}, \quad \frac{a}{\sqrt[n]{b^m}}.$$

Generalization of the concept of an exponent, zero, fractional, and negative. Irrational exponents.

Construction and discussion of the graphs of the functions

$$y = a^x \text{ and } y = \log_a x.$$

Logarithms and the principal properties. Operations with logarithms and antilogarithms. Logarithms on the base 10. Tables of logarithms and their use.

Graphing and discussion of the quadratic functions $y = x^2$ and $y = ax^2$. Graphs and discussion of the functions

$$y = x^2 \pm c, \; y = (x \pm b)^2, \; y = ax^2 + bx + c$$

where a is not equal to zero and where $b^2 - 4ac \neq 0$.

Study of the roots of an equation of the second degree. Geometric interpretation. Decomposition of a trinomial of the second degree into linear factors. Application to geometry.

Theorems on the proportions connected with a circle—properties of the tangent and secant, properties of chords which cut within a circle. The bisector

of an interior angle of a triangle. Numerical relations between the sides of a triangle. Area of a triangle as a function of its sides. Geometric construction of the expressions:

$$x=\sqrt{a^2 \pm b^2},\ x=\frac{ab}{c},\ x=\frac{a^2}{c},\ x=\sqrt{ab}.$$

Solution of equations containing fractions involving the unknowns. Possibility of extraneous roots. Equivalent equations.

Solution of equations in which the unknown is under a radical sign and in which the equation reduces to one of the first or second degree. Possibility of extraneous roots.

Biquadratic equations.

Simplest cases of simultaneous equations of the second degree in two unknowns. Solutions of the types:

(1) $x \pm y = a,\ xy = b.$
(2) $x^2 - y^2 = a,\ x \pm y = b.$
(3) $x^2 + y^2 = a,\ xy = b.$

Position of lines and planes in spaces, perpendicularity, parallelism, angle of inclination between a line and a plane, the theorem of the three perpendiculars.

Relations between planes, parallelism, perpendicularity, dihedral angles.

Angle made by two lines in space.

Polyhedral angles.

Principal properties of parallelepipeds and pyramids, their areas.

Trigonometric functions of an obtuse angle and their relations to those of an angle less than 90° and more than 45°.

Variation and graph of the trigonometric functions of angles between 0° and 180°. Law of sines and of cosines. Their use in the solution of triangles.

Logarithmic tables of trigonometric quantities.

Use of logarithms in all questions studied in geometry, trigonometry, and in algebraic technique.

Ninth Year

Arithmetic and geometric progressions, formulas for the nth term and for the sum. Use of these formulas in solving problems.

Concept of variation, constant and variable quantities.

Concept of infinitely large and infinitely small magnitudes and their limits.

Principal theorems concerning limits.

Increasing and decreasing geometric progressions of an infinite number of terms. Limit of the sum of a decreasing geometric progression.

Irrational numbers defined by two series of rational numbers. Illustration by a graph. Equality and inequality of rational numbers. Operations with irrational numbers. Ratio of incommensurable line segments.

Regular polygons, length of the circumference of a circle, area of a circle.

Cavalieri's Theorem applied to volumes of prisms and pyramids.

Volume of curved solids—cylinder, cone, sphere.

Generalization of the concepts of angle and arc. Angle measurement in radians and in degrees.

Trigonometric functions of the general angle. Graphs of these functions,

their periodicity. Inverse trigonometric functions. Formulas for changing trigonometric functions to simpler form. Functions of the sum and difference of two angles, of a double angle, of a half angle. Formulas with logarithms.

Formulas for the solution of scalene triangles. Computation of the area of a polygon by decomposing it into triangles.

Solution of exceedingly simple equations of higher degree.

Solution of simple exponential and logarithmic equations. Concept of the solution of inequalities of the first degree.

Permutations, arrangements, combinations. Theorem $C_n^m = C_{m-n}^m$. Newton's Binomial Formula $(x + a)^n$ for n a positive integer and the principal properties of the terms of the expansion.

To really comprehend the bearing of this syllabus, it is necessary to discuss again the spirit in which it was made and the purposes which it serves in the teaching of mathematics. The general statements which appear in the *Syllabus and Educational Methods in the Unified Activity School* (Moscow, 1927) give a detailed recapitulation of these purposes. Only the main ideas will be presented.

It begins with the statement that it upholds the same point of view in regard to mathematics and its place in the curriculum as is given in the preliminary memoir of 1925. It then states that mathematics, like science, has no value intrinsic to itself but that it is rather a method of thought. But, it says, if mathematics is not taught for itself and if it does not play even a subsidiary part elsewhere, that does not imply that there is no place for it in the schools. The question has been raised in heated arguments to combat the old idea that mathematics is taught exclusively for one end and that it has a rôle independent of the general system. The place of mathematics in the curriculum may be succinctly formulated as follows: Mathematics is a discipline, practical and necessary for the pupil; an instrument with which he should become familiar for the sake of its uses both in school and in his life outside the school, whatever his calling may be. The utilitarian value in mathematics and the necessity for training in mathematics constitute its justification. Prior to the revolution, the purpose of mathematics, especially geometry, was held to be the training of the mind in logical thinking. Although this idea is recognized to-day, it is not predominant. This neglect of the value of method ought to make itself felt and one must wait for future developments, not denying the utilitarian purposes of the teaching of mathematics but making use of them, as indeed was the case with the proponents of mathematics as training in rigorous thinking.

SCANDINAVIA

By PROFESSOR PAUL HEEGAARD

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Period of Consolidation. Concerning the teaching of mathematics in Scandinavia the years after 1910 may be characterized as forming a period of consolidation in which the great educational reforms of the close of the Nineteenth and the beginning of the Twentieth Century have been realized and tried in practice, and where material for new proposals have been collected. In the year 1928 the preparatory work for such proposals is far advanced in Denmark and Norway; in Sweden they have resulted in the Act of 1927. The changes in the teaching of mathematics, however, will probably not be very significant.

Concerning this period, the interest is chiefly centered in secondary education. Though the three Scandinavian countries as states are mutually independent, they are still connected by strong ties. The causes are partly historical and geographical and partly founded on the affinity of languages. The conformity in the course of development regarding the educational reform is obvious.

Types of Schools. In all the three countries there existed in the nineteenth century three different types of schools without any organic connection: (1) *Latinskolan* (the classical school), preparing for the official career. (2) *Borgerskolan* (the middle-class school). (3) *Folkskolan* (the public school). On account of the reforms referred to above, the first two types were in Denmark and Norway attached to a secondary school, with two successive partitions—the *Middelskolan* and the *Gymnasiet* (the middle school and the *Gymnasium*). The reform was the result of the political growth of the democracy. The final examination of the *Gymnasium* (*Artium*) was mainly meant as a test of the completion of a higher education. But the passing certificate entitles the holder to study at the university. The number profiting by this right is increasing in a surprising manner.

The work of the secondary school in Norway is mainly determined by the act of 1896, further developed by departmental regula-

tions and by supplementary acts of 1897, 1902, 1910, and 1919. (See *Lov om hoejere Almenskoler* (1) *Middelskolen,* (2) *Gymnasiet,* Kristiania, 1911.)

In Denmark the new orientation was created by the Act of 1903. (See *Lov om hoejere Almenskoler m.m.,* Copenhagen, 1907; and the continuations, *Retsregler* (2) 1908, (3) 1912, (4) 1917, (5) 1923.)

In Sweden the reform began with the Act of 1905. (See *Författnings-handbok angaaende rikets allmänna läroverk,* utgiven av B. J:sen Bergquist ock Alfred Nordfelt. Foerra delen, Stockholm, 1910.)

It will not be necessary to enter into details concerning Denmark and Sweden with respect to the time before 1910. It will suffice to refer to the following books:

Der mathematische Unterricht in Schweden, herausg. von Dr. H. von Koch und Dr. E. Göransson. *Der Mathematikunterricht in Dänemark,* Bericht erstattet von Poul Heegaard, Copenhagen, 1912. Rohrberg, *Der mathematische Unterricht in Dänemark,* Leipzig, 1915. Fr. Fabricius Bjerre, *Matematikkens Stilling i den hoejere Skole, Mathematisk Tidskrift A,* Copenhagen, 1927.

For Norway there does not exist any such helpful literature, and so it is desirable to give a short summary. A reform in 1869 in Norway especially joined the middle school and the *Gymnasium.* But entrance to the former was such as to provide for a natural transition from the public school. The tendency in the evolution was to make the complete public school the only prerequisite for the middle school. In the year 1920 this plan was completely realized. In this way the instruction in practical arithmetic in the public school forms a preparation for the teaching of arithmetic and algebra in the middle school.

The organization of the public school is theoretically still based on the Act of 1889, and the training of its teachers on the Act of 1902, but of course many alterations have been made meanwhile. The Minister of Ecclesiastical Affairs and Public Instruction is at the head of this work, as of the other types of schools. The department has to sanction every new schoolbook. This serves to control the evolution of the school work and at the same time affords a certain latitude for private initiative.

Textbooks in Arithmetic. The three most common arithmetics are those of J. Nicolaysen, Ole Johannesen, and Olav Schul-

stad. They treat the following topics: The four fundamental operations with integers, simple fractions and decimals, applications to daily life, interest, discount; areas of parallelograms, triangles, trapezoids, and circles; volumes of solids of corresponding importance; and some work in household accounts and simple bookkeeping.

The oldest book of the three—that of J. Nicolaysen—tended to make the teaching much more clear and intelligible than before. That of Ole Johannesen lays stress on condensed and plain reasoning. The one by Olav Schulstad chooses the problems from the sphere of interest of the children and only such as have practical applications. The book is supplemented by instructions for the teacher. Great stress is laid on the thorough treatment of the elements of arithmetic.

The result of all this movement has been a decided tendency to remove much unnecessary material, including calculations of unused fractions and the finding of curious but useless volumes. The results of the modern investigations into the psychology of the teaching of arithmetic have not yet attracted sufficient attention. However much the ancient methods may be criticized, it still must be conceded that the results in the skill of reckoning were, in general, very satisfactory.

According to the Act of 1896 the middle school had four classes at most, and generally there were only three, admission presupposing familiarity with the number system and with the common units of value, weight, measure, and time, the four operations with integers and decimal fractions, simple practical applications, and skill in mental calculation. Since 1920 there have been generally only three classes.

Objectives. The following teaching objectives are set up:

1. Skill in practical reckoning and its application to the problems of daily life, including the finding of square root, the calculation of areas and volumes, and simple bookkeeping.

2. Arithmetic and algebra, to the theory of exponents, with rational numbers only; the elements of radicals and easy equations of the first degree.

3. Geometry, including the theory of similar triangles, problems in the construction and measure of plane figures, including easier problems about polygons and circles.

The Act of 1902 limits the classes in the *Gymnasium* to three,

and allows pupils to follow any one of several different lines: (*a*) *Reallinjen,* mainly of a mathematical and physical character; and (*b*) linguistic and historical lines with or without Latin.

As objectives for the teaching in mathematics of all kinds there are set up in the Act of 1910 the elements of arithmetic, algebra, and plane geometry in continuing what has been taught in the middle school; the elements of trigonometry; problems in construction and calculation. In *Reallinjen* in addition: advanced algebra; trigonometry continued; the elements of solid geometry; and conic sections analytically treated; the elements of descriptive geometry, with exercises in drawing. For certain parts of the above schedule it is allowed, when desirable, to substitute elementary calculus and its applications.

Two Types of Schools for Teachers. Although the above-mentioned reform of 1896 has to some extent closed the breach between the different types of school, which was the result of former social conditions, the line of demarcation still shows itself between the two types of schools for teachers; namely, the "seminaristic" (for public schools) and the "academic." The systematic pedagogical training of the first of these is the older. At present it is mainly governed by the Act of 1902.

Before the reform of the secondary school, a theoretical university education was regarded as sufficient preparation for the teachers in the *Gymnasium;* but since the reform more and more stress has been laid on the pedagogical side of the university training of teachers. To the academic work in the university (in Oslo) for a degree there has been added a pedagogical seminar with both theoretical instruction and practice teaching. (See *Reglement for den sproglighistoriske og den matematisk-naturvidenskabelige embedseksamen og den paedagogiske eksamen,* 1905, and *Reglement for det paedagogiske seminar og paedagogisk eksamen,* 1908.) Since 1910 this plan of training has been still further developed. The department of public instruction has controlled the development of the plan of training just referred to above partly by regulations, partly by authorization of new textbooks, and partly through the council of education, which inspects the teaching and the examinations.

The plan of education for the secondary school, issued by the department in 1911, recommended the following time schedule for the week:

Subject	Hour-Periods			
	I	II	III	IV
1. For a middle-school with 4 classes:				
All subjects (singing and gymnastics not included)	36	36	36	36
Mathematics, including arithmetic.	6	5	5	5
2. For the *Reallinje* of the *Gymnasium:*				
All subjects (as above)	30	31	31	
Mathematics	5	6	6	
Descriptive geometry		1	1	
3. For the linguistic line of the *Gymnasium:*				
All (as above)	30	30	30	
Mathematics	5	3	0	

The nature of the work tends to discourage the practice of mere memorizing and purposeless formality such as characterized too manifestly the instruction of the past, and to develop a real understanding, self-activity, clearness of expression, and contact with practical life.

This endeavor to form the whole subject of mathematics into a harmonious unit has already led to a more perfect coherence, not only in relation to its different parts, but also in relation to practical life and to the progress of civilization. For example, the teaching of arithmetic in the Class I of the middle school has been so remodeled as to form a more natural preparation for the later work both in arithmetic and in geometry. By object lessons the teacher leads the pupils to an understanding of the fundamental mathematical ideas much more clearly than was formerly the case. This is done by appealing more successfully to the intuition of the learners. In this way the transition to theoretical mathematics with its scientific deductions is rendered more natural and more simple. It has also been found that the way to a liberal education is made more interesting through a moderate use of historical details.

Examinations. The examinations are partly written and partly oral. In the written examinations at the middle school four problems are given, two being in arithmetic (3 hours) and two in mathematics (3 hours), and at the *Gymnasium* three or four problems, partly based upon textbook propositions. It is permitted to

use tables of logarithms, squares, cubes, square roots, cube roots, and the function $\left(1+\frac{x}{100}\right)^n$. The written problems are proposed to all the schools at the same time by a departmental Examination Board.

The Gymnasium. In the year 1919 certain changes were made in the organization of the *Gymnasium,* but these were not of vital importance in the teaching in mathematics. Formerly the division into special courses did not begin until Class II. At present, however, it begins in Class I, where a language course including Greek is introduced. The following is the time schedule:

	I	II	III
	Hours Weekly		
For the *Reallinje:*			
Mathematics	6	5	6
For the linguistic lines:			
Mathematics	5	3	0

For the language courses the amount of mathematics has been somewhat lessened. In the examinations it is permissible to use four-place tables of logarithms of numbers and of trigonometric functions using decimals of a degree instead of minutes and seconds.

Textbooks. As to textbooks, the arithmetics by Ole Johannesen and Fyn-Juel are used in the secondary school. The greater part of the mathematical textbooks (by Bonnevie, Sorensen, Eliassen, Alexander, C. M. Guldberg, Platou and Ole Johannesen) are of the older type; but the textbooks by M. Alfsen (*Plangeometri for middelskolan, Algebra 1 & 2, Plan trigonometri, Elementaer stereometri, Analytisk plangeometri*) are greatly influenced by the modern ideas. They exhibit the development from an abstract formalism to a concrete perspicuity, away from the Euclidean form. It is not sufficient to know *that* a mathematical theorem is right; the pupil should also, as far as possible, know *why* it is so. In order to develop the eye for geometry, the author uses motion as a means of proof (symmetry, rotation). In the work on conic sections a similar plan is followed. The elementary algebra emphasizes the applicability of the subject to practical life and its importance for the economy of thought.

Modern ideas are also seen in a series of textbooks for self-

instruction by Almar Naess. Stress has been laid on clearness of expression. The reader often prepares himself unconsciously for the general theorems, being led inductively by a series of numerical examples.

As already mentioned, the tendency in the development of the teaching of mathematics has been the same in the Scandinavian countries. Nevertheless, the ideas have been more rapidly accepted in Denmark and Sweden than in Norway.

For example, in all three countries the schools are permitted to introduce the elements of calculus, but this permission has been used only in Denmark and Sweden. In these two countries the introduction is now complete and has been well received.

In Denmark the scholarly but somewhat heavy textbooks of Niels Nielsen, Kragh, and Kruger separate strictly the different branches of the subject, materially and methodically. The books of T. Bonnesen issued 1904-09 were the first to embody modern ideas and especially those of Felix Klein, bringing in fusion, early introduction of the concept of functions by graphic representations, and clearer distinctions between axioms and theorems (see Rohrberg, *Der mathematische Unterricht in Dänemark,* pp. 29-32). Many of these ideas are also found in the books of Pihl and Sv. Kristensen (1926-27). The theory of irrational numbers is there given with reference to G. Cantor.

In the upper classes the interest and discussion is concentrated on the books of J. Hjelmslev, *Elementaer geometri* (1) 1916; (2) 1919; (3) 1921; (4) 1923; his *Elementaer aritmetick* (1) 1925; (2) 1926; and his *Den lille geometri* (1) 1925, Copenhagen. His system is based on his distinction between the "geometry of reality" (*Virkelighedsgeometri*) and the "geometry of abstraction." (See, for example, *Die Geometri der Wircklichkeit, Acta mat.,* Vol. 40.) The first is a science of things—edges of rulers, table tops, etc.—considered empirically and inductively, its theorems being only partly proved. The second is a formal science of defined concepts, deductive and precise. Pedagogically the geometry of reality has the advantage of treating of real things, but its theorems lack the simpleness of the abstract geometry. The modern teaching of geometry, with the emphasizing of intuition as a basis, is, properly speaking, a union of the two. As with all textbooks of a revolutionary character, those of Hjelmslev have had difficulties in obtaining access to the schools.

The teaching of mathematics in the language courses has been much discussed in Denmark in recent years. (See *De hoejere Almenskolers Laererforenings Beretning,* 1913, pp. 96-120, and Fr. Fabricius-Bjerre in the *Matem. Tidskrift A.,* 1927, pp. 90-94.) Certain teachers would retain the theoretical mathematics now in use with its exercise in rigorous thinking, while others demand more practical mathematics with exercise in applications to real life. Still others would have no mathematics at all. C. Hansen, who has remodeled the well-known books of Jul. Petersen has, in his book *Anvendt Mathematik* (Copenhagen, 1924), applied mathematics to mechanics and spherical astronomy. The book has been used for several years with good results in the "Metropolitan School." To obtain further experience the department of education in 1924 permitted the schools to substitute for the theoretical mathematics a more practical treatment of the subject.

As in Denmark, the new ideas are already seen in the textbooks of Sweden. This shows itself in the introduction of the function concept in graphical methods, and of tables of logarithms with four figures, and in the effort to secure less heaviness of style. According to information which Dr. Alander has been so kind as to give me, the function idea has been accepted with general satisfaction. The most commonly used books are those of Josephson, Mattson, and Wahlgren. The introduction of graphical methods in algebra has, on the contrary, met with great opposition and partly for this reason books of Wahlgren, Mattson, and Hedström-Rendahl have not been able to supersede the older but excellent work of Möller.

The most widely used textbooks on conic sections are those of Collin and Hedström-Rendahl; and in trigonometry the work of the last two authors is the favorite. In this book the number of formulas has been reduced and the examples are simpler.

In the technical schools in Scandinavia since 1910 there has not been any significant change in the teaching of mathematics except to adapt the organization to the newer features.

One important new development is the training of actuaries. Formerly this work was done by insurance companies themselves, but in 1917 there was introduced an examination of actuaries at the universities of Oslo and Copenhagen, and the same change is proposed in Sweden. The teaching includes besides pure mathematics, theory of probabilities, adjustment, interpolation, statistics, mathematics of insurance, and political economy.

SWITZERLAND

By PROFESSOR S. GAGNEBIN

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Uniformity of Instruction. In the report on the teaching of mathematics in Switzerland presented at the International Congress in Rome in April, 1908, Professor Fehr stated that the most noteworthy characteristic of the organization of the Swiss schools lay in the academic autonomy of the twenty-five states (cantons and half-cantons [1]) which make up the Swiss Confederation. One can easily imagine the difficulties that had to be met in establishing a real uniformity among the varied programs of the *gymnases* of the cantons and communes. The struggle to achieve this result lasted for nearly sixty years, the decisive years falling within the decades with which we are principally concerned. Two factors have made major contributions to the establishing of uniformity of instruction. The first of these was the founding of the *Ecole polytechnique fédérale* in 1855.[2] The second was the adopting of the Federal Constitution of 1874, which gave the Federal government authority to regulate the conditions which must be satisfied by those who wished to practice medicine in Switzerland. At the same time, the Constitution empowered the state to intervene in the training of medical students.

The Final Step. The final step in the centralization of control was an act called "A Law on the Recognition of Graduation Certificates by the Federal Board" (January 20, 1925).[3] (These graduation certificates are the degree of *baccalauréat* to which later reference will be made. This degree is granted at the completion of the thirteenth school year.) Prior to the passing of this bill, two independent authorities were concerned with the examinations: the

[1] Three of the twenty-two cantons or states of Switzerland are divided into half-cantons whose power in local affairs is equivalent to that of a whole canton but whose share in federal matters is half that of a canton.—Editor.

[2] This school, located at Zürich, is as its name indicates a federal institute of technology offering advanced work in many fields.

[3] *Ordonnance sur la reconnaissance des certificates de maturité par le Conseil fédéral suisse.*

Federal Council of the *Ecole polytechnique fédérale* and the Federal Commission on *Baccalauréats.*

Regulations of 1908. The most recent regulations concerning admission to the *Ecole polytechnique fédérale* is of the date November 7, 1908. It contains an examination syllabus and provides for an agreement between the Council of the *Ecole polytechnique* and the schools which granted the *baccalauréat.*

Regulations for Medical Examination. The regulations for admission to the state examinations for the medical professions may be summarized as follows: a decree of the Federal Board dated March 10, 1891, creating a Federal Commission on the *Baccalauréat* (*Commission fédérale de maturité*). The last ruling of this commission was dated July 6, 1906. It contained an examination syllabus, but the Commission could not issue certificates except to candidates who satisfied the necessary conditions for passing the examinations for the *baccalauréat* in a Swiss *gymnase.* On the other hand, it listed schools whose graduation certificate was recognized as a *certificat de maturité,* and the commission was under obligation to assure itself from time to time that the schools here listed were continuing to offer the various courses required by the regulations.

Three Types of Certificat de Maturité. By the Act of January 20, 1925, the Federal Board recognized three types of *certificats de maturité:* (A) the Greek-Latin type, (B) the Latin-modern language type, and (C) the mathematics-natural science type.

These three types of diplomas carried with them the privilege of admission to the state examinations for analytical chemists and of admission as a regular student without other examination to the first semester of the various divisions of the *Ecole polytechnique fédérale.* Candidates not possessing one of these three diplomas were required to pass an entrance examination. From the point of view in which we are interested, this syllabus is almost identical with that which leads to the diploma of type C (mathematics-science). Diplomas of types A and B carry the right of admission to the state examinations for the medical profession. The same is true of the diploma of type C when the applicant has passed an additional examination in Latin before the *Commission fédérale de maturité.*

This Commission proposed that the Federal Board recognize the degree of *baccalauréat* granted by a canton, provided that the re-

quirements for this degree should satisfy the conditions imposed in the Ordinance and provided that the canton should guarantee that the school issuing the degree continued to satisfy the stipulated conditions.

Thus the admission to the government examinations rested in the hands of two authorities. Now there is but one. The importance of this is self-evident as a step in the unifying of the curricula of the *gymnases* and as a possible consequence, in unifying that of the Swiss universities.

The recapitulation of the issues involved and of the debates which resulted in the publication of this ordinance by the Federal Board would exceed the limits of this report.[4] I will merely say that the syllabi that accompany the ordinance may be considered as the result of a compromise concerning three things: the demands of the *Ecole polytechnique fédérale;* the demands of the *Comité directeur des examens de médecine* and the whole medical staff; and the desires of the school authorities who were defending the autonomy of the institutions granting the *baccalauréat.* This syllabus is also the result of the general spirit which manifested itself among us during the World War and which is characterized by a tendency to set a high value on what is customarily called personality or general culture rather than on a great mass of special knowledge. Article 15 of the Ordinance is a pledge of this. It deals with the maturity of attitude required by the higher schools and it closes with these words: "The development of the qualities of the mind, the training of the will and of the character quite as much as physical education should keep step with the development of intellectual maturity."

This tendency is very evident also if one compares the two examination syllabi for admission to the *Ecole polytechnique fédérale.* Although the ruling of November 7, 1908, provided for but one written examination—for general background, an essay in French, German, Italian, or English—the ruling of July 23, 1927, requires two; that is to say, an essay in the mother tongue and a question in a foreign language. Furthermore, the part of the syllabus relating to general culture occupies one page in the first of these regulations and but two in the second; whereas the sections relating

[4] For this, consult *Les Collèges et les Gymnases de la Suisse,* by Dr. A. Barth, Dean of the *École de jeunes-filles* at Basle. French edition by Ch. Gillard, Director of the *Gymnase classique* in Lausanne.

to special subjects have kept the same dimensions as before. In other respects, the examination for admission is practically equivalent to the *certificat de maturité* of type C (mathematics-natural science course).

Before concluding this topic, it should be repeated that the two other courses (Greek-Latin and Latin-modern language) also prepare students to enter the *Ecole polytechnique.*

Effect on the Teaching of Mathematics. Let us now consider the bearing of this general movement on the teaching of mathematics. From this point of view, let us make a brief comparison of the syllabus for admission to the *Ecole polytechnique* as it appeared in 1908 and again in 1927. The principal difference is that the function idea is now greatly stressed. Its introduction is prepared for by a study of the graphic solution of equations of the first and second degree. Later, the student considers the graphic representation of the functional relation of two quantities chosen from the field of mechanics or of physics. He then goes on to the derivatives of rational and simple transcendental functions, and finally he applies these concepts to the study of the variation of functions (*l'étude des variations des fonctions*). Another innovation is the application of the study of combinations to simple problems in probability and in life insurance in the same way as spherical trigonometry is applied to mathematical geography.

Geometry. Geometry is presented as the study of the properties of space: coincidence by displacement, similarity, symmetry, geometry of position (*relation de position*), and geometric constructions. The new program also includes the study of poles and polars, and it requires practice in geometric design in pencil and in water colors.

Algebra. On the other hand, the course in algebra no longer includes the algebraic and trigonometric solutions of equations of the third degree, nor does it include the study of the properties of regular polygons from the point of view of the division of an arc. Moreover, it lessens the work in the elementary concepts of a series.

These changes may be described as a concentration of the syllabus about the fundamental principles: practice in computation which is of paramount importance, knowledge of the geometric properties of space, and the acquisition of the analytic and graphic tools that permit the pupil to study the functions and to represent them.

We should remember that in modifying the curriculum of the French *lycées,* the regulations of 1902 and of 1905 showed the influence of this objective. Professor Fehr sponsored the same aim in an address before the Swiss Society of the Teachers of Mathematics in 1904. In this address, he called attention to the fact that Felix Klein had presented the same views in the vacation lectures at Göttingen at Easter in the same year.

Similar remarks might have been made in regard to the syllabus in mathematics in the *Règlement des les examens de maturité pour les candidats aux professions médicales* (July 6, 1906), and in the *Règlement pour les examens du type A et B* of 1925. The second of these is built about the function idea, but the study of functions is not carried so far as it is for the certificate of type C which we are about to consider.

On the other hand, the new program omits both the solution of equations of the second degree in several unknowns and the whole topic of combinations (*analyse combinatoire*). Descriptive geometry does not appear in the curriculum leading to the certificates of types A and B.

The Syllabus in Mathematics and Physics. The syllabus in mathematics and physics is as follows:

Mathematics

Theory of numbers (*arithmetique*), algebra, and analysis: Concepts of rational and irrational numbers. Algebraic computation. Logarithms. Equations of the first degree in one or more unknowns. Equations of the second degree in one unknown, algebraic and graphic solutions. Arithmetic and geometric progressions. Compound interest and annuities. Ratio of dependence and the graphic representation of functions.

Geometry: Simple geometric forms. Relative position and constructions in a plane and in space. Coincidence, similarity and symmetry. Simple methods of drawing. Computation of areas and volumes.

Trigonometry: Right triangle. Law of sines and law of cosines, for the general triangle. Solution of triangles. Trigonometric functions of various angles and the addition theorems of these angles.

Analytic Geometry: Study of the point, straight line, and circle by rectangular coördinates. Simplest equations of conic sections. Principal properties of these curves.

ADDITIONAL REQUIREMENTS FOR COURSE C

Complex numbers and operations with these numbers. Second degree equations in two unknowns. Approximate solution of equations. Elements of the theory of combinations. Simple problems in the calculation of probabilities and life insurance. Derivatives of rational functions and of the simplest

transcendental functions. Approximations to the length of arcs and to surfaces and volumes.

Goniometry. The general plane triangle. Right spherical triangles. Law of sines and of cosines for the general spherical triangle. Applications taken from mathematical geography and astronomy. The study of conic sections by means of poles and polars.

DESCRIPTIVE GEOMETRY (TYPE C)

Representation in a plane and in elevation of a point, line, and plane, and the constructions relating to them. Plane figures drawn in projection and isometrically. Drawing of polyhedra, plane sections and intersections. Drawing of cylinders and of right cones. Study of their points, generatrices, tangent planes, and cross sections. Representation of a sphere. Geometric design. Constructions with ruler and compass. Use of pencil and water colors.

PHYSICS

Fundamental concepts of mechanics. Equilibrium of solids. Wave theory. Production and propagation of sound. Fundamental ideas of acoustics as applied to music. Heat. Expansion due to heat. Measurement of heat. Elements of thermodynamics. Change of state due to heat. Optics. Straight line propagation of light, reflection and refraction. Measurement of light. Dispersion. Optical instruments. Spectrum analysis.

Magnetism. Electrostatics. Electric currents. Magnetization of solids, liquids, and gases. Practical methods of measuring electricity. Heating effect of an electric current. Current through a circuit. Induction. Additional requirement for type C: dynamics, elements of physical optics.

Newer Developments. It is evident that the syllabus of 1925 showed a reduction in the required subject matter, but this reduction was in harmony with the remaking of the syllabus on a new plan; namely, in centering it about fundamental concepts, which was the desire of those who had been occupied with the teaching of mathematics in the preceding years. In debates such as those raised by the issue of federal certificates, from which politics has not always been absent, the type of instruction such as that with which we are concerned is put on the defensive; and on the whole, we have cause to congratulate ourselves on the outcome. The Swiss Society of the Teachers of Mathematics founded in 1901 certainly played its part. It was able to keep constantly in touch with the representatives of the *Conseil de l'Ecole polytechnique*. At the same time, the members of this board always displayed great interest in questions regarding instruction in the secondary schools. They were present at the meetings of the Society and even presented important papers at them. It is there that Professor Meissner read his *Report on the Teaching of Mechanics in Secondary Schools*[5]

[5] *"Rapport sur l'Enseignement de la mécanique à l'école moyenne," Revue générale des Sciences pures et appliquées*, December 31, 1927, p. 689.

which was so valued by his hearers, and which received so much attention in France. In connection with the new regulations for admission to the *Ecole polytechnique,* Professor J. Franel presented a work which was published in his *Annuaire* and which has undoubtedly contributed to the establishing of new bonds between this school and the teachers of the secondary institutions.

Influence of the Swiss Society. The Swiss Society of Teachers of Mathematics has constantly stimulated work on divers points of the program of instruction in the *gymnases.* Finally, in a conference at Zoug (1922) it fixed upon elaboration of the plan of study which merits particular attention from the point of view of the question raised at the beginning of this report. We should call to mind the important publications undertaken by the Swiss Subcommittee of the International Commission on the Teaching of Mathematics (1908-20). These formed a complete picture of the work in Switzerland in all levels of instruction. Of the nine sections of this report which appeared under the direction of Professor H. Fehr, one was devoted to the Swiss *gymnases,* classic as well as scientific. This was written by Professor K. Brandenberger of Zürich, whose memory is greatly revered by the mathematics teachers of Switzerland.[6] The author uses very ingenious and complete tables to show first that there is no uniformity of curricula in *gymnases;* second, that the syllabus for admission to the *Ecole polytechnique* (1908) and that of the *maturité fédérale* in preparation for the medical professions (1906) have always been considered as minimal requirements and that these requirements have been greatly exceeded in the secondary schools; and third, that the materials treated in these schools are exceedingly diverse and in the majority of cases are much more numerous than those which appear in the federal syllabus.

Plan of Study. Several times in the meetings of the Society of the Teachers of Mathematics a desire was expressed of framing a plan of study which might be consulted on the occasion of changes in the curriculum in the *gymnases.* This was one of the theses with which the late Professor Otti of Aarau concluded a memoir presented at Baden in October, 1915, on the subject: "What topics may be omitted in the program of instruction in mathematics?"[7] This

[6] This is an octavo booklet of 167 pages published by Georg, Basle and Geneva, 1911.

[7] *"Quels chapitres pourrait-on supprimer des programmes d'enseignement des mathématiques?" L'Enseignement mathématique,* Vol. 18 (1916), p. 138.

also is Conclusion 8 of the report which the Federal Department of the Interior asked of Dean Barth of Basle and which appeared in 1919.[8]

This plan of study in mathematics was developed by a commission under the presidency of Dr. H. Stohler of Basle, the editorial work being entrusted to Dean Amberg for types A and B, and to Professor Schüepp for type C. This plan of study was submitted to the members of the Society in January, 1926. It remains substantially within the limits set by the *Ordonnance* of 1925, scarcely exceeding these boundaries, but it is much more detailed, the topics for instruction are arranged with great care, and it is accompanied by suggestions of teaching methods and by explicit statements of the point of view which facilitates the student's learning. The plan indicates the number of hours that should normally be devoted to the study of mathematics in each class. Lastly, it includes a table compiled by Professor Flükiger of Berne, giving the average number of hours devoted to the various branches of mathematics on each level of instruction in December, 1925.

Publication of Teaching Manuals. I should also add that the Society of Teachers of Mathematics and the Swiss Mathematical Society had a special joint meeting at Berne on May 20, 1928. It was decided at this meeting to publish teaching manuals which should conform to this plan of study. An editor was chosen and an editorial committee was selected for each part of this work. Each section is in two volumes—one dealing with theory, the other consisting of exercises, applications, and problems.

These works are written in German. The writer may perhaps be permitted to express the wish that the teachers of mathematics undertake a similar publication in French. This is primarily a question of finances, but it should be noted that a representative of French Switzerland, Professor Charles Jaccotet of Lausanne, was a member of the committee that worked on the German manuals.

New State of Affairs. It is evident that this statement of Professor Brandenberger, which applied so perfectly to our situation in 1911, is no longer precisely true. The march of events and the energy of certain members of the Swiss Society of the Teachers of Mathematics have brought us to a new state of affairs. By a concerted effort, the teaching of mathematics has been unified, it

[8] See the French edition by Gillard referred to on p. 122.

has been better adapted to new needs, and it has been given a guiding thread due to the experiences of numerous teachers in Switzerland. We certainly owe a great debt of gratitude in this regard to the International Commission on the Teaching of Mathematics, through whom we have learned so much of what is going on in the different countries in Europe, in the two Americas, in Australia, and in Asia. I should cite here the name of Dr. H. Fehr, professor at the University of Geneva, secretary-general of the International Commission on the Teaching of Mathematics, now dissolved, president of the Swiss subcommittee, and director of *L'Enseignement mathématique,* the official organ of the International Commission. By his works, his publications, and his correspondence, he has made a great contribution, the results of which we enjoy to-day.

Examination Problems. In order that this report on the teaching of mathematics shall not be purely a historical summary, and in order that the reader may form his own opinion of the level of the studies in one of our secondary schools, I shall quote the problems set by Professor L. Gaberel for the candidates for the *baccalauréat* in sciences in the *Gymnase cantonal* of Neuchâtel in July, 1928. These problems cover the knowledge of the requirement in geometry, analytic geometry, algebra including the solution of equations of the third degree, and the calculus including the idea of a partial derivative.

1. Let A be a given point on the axis of a parabola, required to draw a chord CD which shall be perpendicular to the axis at a point B which shall lie between A and the vertex O, such that the volume of the cone generated by the triangle ABC in its rotation about the axis shall be a maximum.

2. A right circular cone is inscribed in a sphere of radius r. What shall be the ratio of the height of the cone to the radius of the sphere in order that the ratio of the volume of the cone to the volume of the sphere shall be as l is to n? What is the value of the ratio if n is $\frac{27}{8}$?

3. Change the equation of the conic $2x^2 - 3xy + 3y^2 + x - 7y + 1 = 0$ to canonical form.

Several candidates gave solutions for all three problems. Besides these, they were required to make a descriptive geometry diagram and to solve three problems from mechanics dealing with statics of solid bodies, and the kinematics and dynamics of a point.

Preparation of Teachers. The question of the preparation of

teachers of mathematics in the secondary schools, which has been so widely discussed in all countries in the last twenty years, has come up for consideration in Switzerland also. As early as 1906, the Swiss Society of Teachers of Mathematics expressed the desire to study the question. The important report of Gutzmer and Klein given at the conference in Dresden appeared in *L'Enseignement mathématique* (1908, pp. 1-40). In 1915, the same review published the International Commission's questionnaire on this subject in four languages and Professor Fehr presented this questionnaire to the Society of Teachers of Mathematics. Finally in 1917, the Society approved a work of Professor K. Matter, as a result of which it proposed a vote recommending the formation of university courses dealing with (1) questions of elementary mathematics considered from the point of view of higher mathematics and (2) the history of mathematics and the study of this science from the point of view of the theory of knowledge. It also asked for the creation of special institutes for the theoretical and practical study of the teaching of mathematics.

In response to these needs, universities such as those of Basle, Geneva, Zürich, Berne, and the *Ecole polytechnique* several years ago organized courses dealing with the teaching of mathematics. The University of Lusanne, that of Neuchâtel, and others besides, offer courses in the history of science. For some time the Canton of Vaud has required a teaching certificate from candidates for positions in the secondary schools and several other cantons have followed their example. All the universities now issue teachers' diplomas (*certificats pédagogiques*) to those who qualify on the basis of practical work.

An effort is being made in several of the Swiss universities toward a greater concentration of the curriculum. I will cite only the example of the Faculties of Science of the Universities of Lausanne and of Neuchâtel where, several years ago, courses were organized leading to teachers' diplomas somewhat after the example of those in France. The university degree is required for admission to teaching in secondary schools in French Switzerland. Heretofore, the examination for the degree was very comprehensive. The new requirements allow the candidate to concentrate his attention on the most important branches of mathematics and on its allied sciences. It also allows him a wide choice among a great number of combinations of topics which are so arranged, however, as to insure

that the mathematical training of the candidate shall be well organized.

New Swiss Journal. Lastly, if it is true that a good teacher should always keep in touch with the part of his science that is in the making, a report on the teaching of mathematics should not fail to mention the Swiss periodical devoted to mathematics. The Mathematical Society of Switzerland decided to found such a journal at its joint meeting with the Swiss Association of Teachers of Mathematics on the twentieth of May, 1928. The editorial work will be in the hands of Professor Füeter of Zürich assisted by an advisory board, and matters concerning articles in French will be entrusted to Professor Juvet of Neuchâtel. The first number of this journal is that of November, 1928, less than a year after the appearance of the first number of the *Helvetica physica acta.* The mathematics journal will bear the name *Comentarii Mathematici helvetici.* I could not close my report with mention of an event more full of promise for my country.

UNITED STATES

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I. Introduction

In Retrospect. From 1910 up to the present time everyone interested in mathematics in the United States has been trying to discover better methods of organizing and teaching the subject. This has been clearly shown by such movements as that initiated by John Perry in England, commonly known as the *Perry Movement* or *The Laboratory Method.* The influence of this great pioneer on our thinking should not be overlooked.

Causal Conditions. In order, therefore, to understand fully the significance of the changes that have taken place in the teaching of mathematics in this country since 1910 and the trends that now exist, it will be helpful to consider some of the causal conditions existing back as early as 1910, even though we cannot take the time here to discuss each fully. As pointed out by Professor Smith [1] in discussing these general conditions:

> At the beginning of the present century the syllabi in mathematics in the American high schools were determined largely by the requirements for entering our colleges. As a rule examinations were set by each college for its own candidates, the requirements being dictated by the department of mathematics.

College Entrance Examination Board. The College Entrance Examination Board was organized in 1900, and while this was a forward step the influence of tradition still forced the retention in mathematics courses of a great deal of material which we now consider obsolete and which has actually been removed from the present College Entrance Examination Board Syllabi.

Committee of the American Mathematical Society. A committee of the American Mathematical Society in 1902 also formulated a report on "definitions of college entrance requirements,"

[1] Smith, David Eugene, "A General Survey of the Progress of Mathematics in Our High Schools in the Last Twenty-five Years," *First Yearbook of the National Council of Teachers of Mathematics*, 1926, p. 1.

making certain specific recommendations with reference to the content of algebra, plane and solid geometry, and trigonometry. Speaking of this report, Professor Smith [2] says:

> This report was evidently rather inclusive through its very lack of precision. It kept open the way for every eccentric examiner to propose almost any question he wished and yet it served fairly well as a starting point for reform. At any rate, it was the expression of a national instead of a local opinion.

While the work of the College Entrance Examination Board "sought to unify the examinations and prepare them with much greater care than was usually the case with local efforts, it also gave an opportunity for the schools to be consulted and become a part of a central organization, thus being represented in the preparation of the papers." Although the work of this board influenced by the committee of the American Mathematical Society has been considered by many an improvement over the situation before 1900, there are still some objections to the system *as such* in certain quarters.

Limitations of Examinations. College entrance examinations grew out of a desire to standardize the mathematical product of the schools. The result of leaving each school to determine what a pupil should know in order to enter any higher institution of learning was often chaotic. However, the results have sometimes been detrimental to the best interests of mathematics. Teachers should be encouraged to have a philosophy of their own and to teach the subject as it ought to be taught rather than to try to prepare their pupils for *one final examination*. This they will not do so long as they are forced to follow a course of study which they have had no hand in making or with which they are not in sympathy. No matter how desirable a curriculum may be, we cannot expect it to succeed if the classroom teachers are not given a chance to help in its construction. That is why curriculum construction should be dynamic—a training of teachers in service.

Tradition has been a hard factor to overcome in modernizing the curriculum in mathematics, the difficulty being largely a matter of clinging to the hazy and invalid objectives used in teaching the mathematics of many generations ago. We shall discuss this more fully later.

In all fairness to the College Entrance Examination Board it

[2] Smith, David Eugene, *op. cit.*, p. 2.

must be said that their attempts to improve the situation have resulted in syllabi in mathematics more desirable than those in use in many schools.

Other Examinations. Certain other examinations have influenced the offering in mathematics, particularly those of the New York Regents system which, until recently, has been more conservative than those of the College Entrance Examination Board. In 1910 extra-mural examinations were cited as the reason why more progress could not be made. Similar reasons for lack of progress are being given to-day in such schools as are able to offer little to their students except what is considered proper teaching material by extra-mural examining boards.

Effect of Examinations. In many places throughout the country teachers of mathematics have argued that their teaching has not been influenced by the standards set up by extra-mural boards. However, when one stops to reflect that most of the textbooks are written to meet the demands of certain syllabi, and that in most schools the textbook is the course of study, it is certain that such syllabi still exert great influence on the content of our courses in mathematics. As a result, extra-mural examinations still tend to reduce the initiative of the teacher and in other ways leave conditions far from satisfactory in many schools. The trouble is not that the extra-mural syllabi are obsolete—it is that too many teachers become slaves to the system in vogue.

Mathematics Report of the N.C.A. Let us consider especially the *Report of the Committee on Mathematics of the North Central Association of Colleges and Secondary Schools* [3] (hereafter referred to as the N.C.A.).

One is inclined to look upon this report as chiefly the work of its Chairman, Professor E. H. Moore, of the University of Chicago, because of its similarity to his famous presidential address of December, 1902.[4] In this report emphasis is given to many advanced points of view, among which are the following:

1. Algebra should be taught as generalized arithmetic.
2. Geometric forms should be taught with arithmetic.
3. Intuitive geometry should be introduced.
4. Literal representation of numbers should be gradually introduced.
5. One should go from the concrete to the abstract.

[3] See *Proceedings of the N. C. A.*, 1903, pp. 138-39.

[4] Moore, E. H., "On the Foundation of Mathematics," *First Yearbook of the National Council of Teachers of Mathematics*, pp. 32-57.

6. Pupils should be trained (*a*) to think and observe accurately; (*b*) to describe things accurately in language, in picture, and by means of the equation; (*c*) to infer correctly and to act on inference; (*d*) to formulate clear statements of what they have done.
7. The pupil should be an active worker, not a passive listener.

A report of the N.C.A. was given in 1908, but it was largely a repetition and reinforcement of the report of 1903; we shall not discuss it here.

It is interesting to compare the 1903 report with the 1910 report of the N.C.A.[5] The latter contains the following suggestions:

1. Omit complicated forms of parentheses, fractions, factoring, and equations.
2. Algebra should be training in thinking rather than manipulation of complicated expressions.
3. The truth of many geometric theorems should be accepted without proof.
4. Originals are of utmost importance.
5. Exercises in geometry involving algebra and numerical applications should be encouraged.
6. Confidence in one's own power of correct reasoning, and the ability to discover geometric relations are of more importance than the ability to recall the demonstration of a large number of theorems.
7. Omit the theory of limits.
8. The aim is to develop and strengthen the ability to perceive exact relations and to make inferences correctly.
9. "The teacher's constant aim should be to train the pupil to think, and to formulate clearly the results of his thinking."

Influence of College Professors in the N.C.A. Prior to 1903 and even later the situation in mathematics was controlled largely by the college professors of pure mathematics, and their opinions were law and gospel for all concerned. The following table shows the numbers of each group in the N.C.A. from 1906 to 1910:

Membership in the N. C. A.	1906	1908	1910
College Professors of Mathematics	10	10	11
College Administrators	1	2	2
High School Principals	3	4	5
Heads of Mathematics Departments in High Schools	2	2	1
High School Mathematics Teachers	2	1	3
Total	18	19	22

[5] *Proceedings of the N.C.A.*, 1910, pp. 84-87.

It was generally agreed that English and mathematics should be constants in the curriculum, and at one time in the N.C.A. a resolution was made and defeated that six years of mathematics should be required.

In spite of the fact that the college professors of pure mathematics held the balance of power in these early days, they were interested in method and in general improvement.

Practical Mathematics. In 1910 agitation for courses in practical mathematics was a common thing. The final report of the National Committee of Fifteen on a geometry syllabus [6] gave strong emphasis to the need for problems in architecture, design, physics, and mechanics, and made various other attempts to introduce genuine problems. Vocational and utilitarian mathematics were at their "peak" during the next few years. In 1929 the same kind of agitation as that of 1910 is quite prevalent.

II. Influences Since 1910

Types of Influence. The actual trends in the teaching of mathematics to-day can best be discovered by a statistical study of the present practices in books and methods. Time and expense make a study of these practices questionable, and what is more vital, the significance of such a tabulation is in itself unimportant. We do not care in particular what is being done in general, but we are whole-heartedly interested in the indicated or future trends as we see them functioning in progressive communities, or as recorded in articles by frontier thinkers.

We shall consider three types of influence that are important for this discussion; namely, depleting influences, enriching and widening influences, and organizing influences.

The last two especially are more or less direct outcomes of the shift in emphasis in education from subject matter to children's interests, capacities, and needs.

I. DEPLETING INFLUENCES

The Power of Tradition. In addition to the depleting influence of extra-mural examining boards already discussed, let us examine some of the other factors that still hamper progress. Undoubtedly the influence of tradition or the feeling of satisfaction

[6] "Final Report of the National Committee of Fifteen on Geometry Syllabus," *The Mathematics Teacher,* 5:46-131.

with the *status quo* has retarded improvement in our courses in mathematics more than anything else. As one writer puts it, "Tradition has become a serious and deadly opponent to progress in mathematics."

Age to Be Respected. What has been said above is no *a priori* argument against any of the time-honored content material or methods of presentation that can be justified on any rational basis. The fact that any topic or method has stood the test for ages is some reason for respecting it, but it does not mean that age alone gives one precedence over the other, or that it should become sacred. Professor Nutt says: [7]

> Teachers and supervisors are inclined to think in terms of the subject instead of in terms of the student. Subject matter has been standardized instead of stages of mental maturity of students. The teacher and the supervisor have been dealing with the subject so long that it has become a familiar acquaintance; hence it has become more or less a sacred thing. The subject has become a habit with them; it is regarded as something permanent and abiding; hence to leave out any of the sacred facts seems almost sacrilegious and criminal. On the other hand, the student is transient. Students come and students go; hence to leave the student out is justifiable. In fact, leaving the student out may be getting rid of an unappreciative butcher who haggles and mangles the sacred subject most horribly in his attempts to find food for mental maturing. The relief that is usually manifested by teachers and supervisors when a student, who is not getting on, drops out is a definite indication that the subject is more important to them than is the student. Whenever teachers and supervisors begin talking about education by means of the subject "getting on" in the student instead of the student "getting on" in the subject, then a radical change will come about in the teaching in secondary schools.

Domination of the College. Closely allied to the influence of tradition in retarding growth is the influence of the liberal arts college, at least in many parts of the country. Speaking recently of this situation Professor Frasier [8] said:

> The present domination of the colleges over the high schools would be a good thing for the high schools if these colleges had a conception of education as something to do with modern life. But the average college of this group still clings to its medieval curriculum.
>
> If one is to judge from college entrance requirements, the liberal arts colleges believe that to be a real "Liberal" one must be a worshiper of the past. The greater he is in the circle the lower he kneels before the shrine of things old.

[7] Nutt, H. W., *The Supervision of Instruction,* p. 53. Houghton Mifflin, 1920.

[8] Frasier, G. W., "The Responsibility of Higher Institutions of Learning for the Development of American Education," *Teachers College Record,* 30, p. 115.

The Theory of Mental Discipline. Another influence which affected the aims of instruction a generation ago was the theory of mental discipline. As Professor Thorndike points out, the theory was "rhetorically attractive." While this theory held sway one kind of algebra, for example, was thought to be as good as another in the education of youths. As Mr. Dooley would probably say, "The only thing we need to do is to make mathematics hard so that children will hate it. The more they hate it the better it will be for them." With the decline of what has now come to be regarded as the mistaken claims and interpretations of this theory on the part of overzealous mathematicians there has come a more careful consideration of the practical and cultural opportunities afforded by certain subjects. Thus, the general aim has shifted from the mental discipline point of view to that of giving the learner something that can be interpreted readily as a contribution to a broader, more useful, and richer life. Still more significant is the plan of stimulating and securing the interest of the pupil in his own activities and welfare.

Not only the psychologists and laymen but teachers of mathematics as well have realized that this reason for teaching mathematics has been carried too far and has led to unfortunate conditions with reference both to content material and to methods of teaching.[9]

Transfer Value of Mathematics. It was formerly believed that mathematics should be studied because the habits of logical thinking, of precise and accurate work, thus engendered, would aid the pupil in other subjects studied as well as establish certain valuable life habits. Teachers have failed to see that except for the more gifted pupils transfer will not take place unless the work is properly developed. Moreover, we shall see that a very decided movement has been set on foot looking toward the simplification of subject matter. This has been particularly true where intuitive geometry has been introduced into the seventh and eighth grades of the junior high school. It is also true where algebra has been entirely reorganized so as to modify radically the traditional method of approach without losing the transfer value.[10]

[9] Betz, Wm., "The Teaching of Geometry in Relation to the Present Educational Trend," *School Science and Mathematics,* 8 : 625-33. See also 9, pp. 494 and 789.

[10] Millis, J. F., "Some Modern Notions in the Rational Teaching of Elementary Algebra," *School Science and Mathematics,* 7 : 176-82. See also 7 : 70-72 and pp. 64, 110, 517.

Minimum Essentials. The practice of preparing a list of so-called "minimum essentials" has proved to be worth while in schools, provided these essentials have not become the standard for all—a standard of mediocrity actually existing to-day in many places and resulting in making our gifted pupils the most retarded of all. Fortunately, we are coming to see that lists of objectives must include for the brilliant pupils those which go far beyond the needs of the modal child. It may be all right to start with what the modal child can be expected to do successfully, but we should consider the needs of the slower and of the more gifted children as well.

Mathematics as a "Tool Subject." We have been hampered in our attempt to provide a fuller and more complete education for the American citizen by the attitude of those who look upon subjects as "tool subjects." Take arithmetic,[11] for example. Professor Judd says: [12]

> I can hardly expect to stem the tide of common opinion by anything that I can say in a single paper, but I am here to urge that the term "tool subject" be carefully reconsidered. For my own part, I reject it absolutely. The experiences which have come into modern life from the study of number are not the trivial rules of addition and subtraction and the rest; they are experiences of a wholly different order. The curriculum maker who thinks that he has exhausted the catalogue of uses of number when he has listed the examples which ordinary men solve in a day or a week is superficial to such an extreme degree that he is an unsafe guide in arranging the plans of the school. The man who calls arithmetic a "tool subject" and with this name dismisses it as something less worthy than subject matter courses is guilty of criminal neglect of true values.

Emotionalized Attitudes in Learning. We need to give more attention to the influence of emotionalized attitudes in learning. Professor Briggs [13] has given a very interesting discussion of this important question. We should know more about how such attitudes are developed and how they may reinforce our teaching. At the present time too many people dislike the study of mathematics. I believe that this is unnatural. Perhaps the trouble goes as far back as the arithmetic of the elementary school. Certainly the teacher at this point in the pupil's mathematical development has a great responsibility because of the importance of first impressions. There is some doubt whether all pupils should be encouraged

[11] Wilson, G. M., *What Arithmetic Shall We Teach?* p. 1. Houghton Mifflin, 1926.

[12] Judd, C. H., "The Fallacy of Treating School Subjects as 'Tool Subjects,'" *Third Yearbook of the National Council of Teachers of Mathematics*, pp. 3-4.

[13] Briggs, T. H., *Curriculum Problems*. Macmillan, 1926.

and stimulated to continue the study of mathematics beyond the junior high school. But whether they continue or not, they should not be taught to hate the subject.

Failures. We have too many failures in ninth-grade algebra and the pupils spend too much time failing.[14] We cannot justify failures of from twenty to forty per cent in a normal group in any school. Are there any pupils in such a group "who simply can't learn anything in algebra," or who, unable to learn a certain kind of algebra, fail simply because the teacher does not know what else to do? Or is it possible that some of our seemingly dull pupils may do creditable work if they can only be properly aroused? Many failures are due to the lack of recognition of individual differences in ability and large numbers of deserving pupils have to pay the penalty. It is clear that our entire system of marking needs further investigation, and that we need a new method of marking which will do justice to those pupils who ought to pay big returns on the capital invested. In attempting to help the duller pupils we can learn a great deal about the difficulties which pupils generally encounter. In this way instruction even for the gifted pupils may be improved.

Standardized Tests. In contrast to the old essay-type of examination, a standardized test is an example of one kind of new-type test in which some standard performance of so many questions correctly answered in a certain number of minutes has been worked out.

No one who has followed the testing movement will question the fact that standardized tests have made certain contributions to education.[15] On the other hand, careful students of education are aware of three points of attack upon these newer instruments of measurement. I refer to the careless construction of many of the tests, their misuse in many classrooms, and the faulty interpretations often made as a result of their use.[16]

A steamboat should not be condemned because it is not a Packard or even a Ford. It was not so intended. In like manner, standardized tests are not to be condemned if they serve adequately the purposes for which they are constructed. Where standardized

[14] Flexner, Abraham, *The Modern School,* Occasional Papers, No. 3, General Education Board.

[15] Reeve, W. D., "Educational Tests—To Standardize or Not to Standardize," *The Mathematics Teacher,* 21 : 369-70.

[16] *Ibid.,* pp. 370-77.

tests are valid, objective, and reliable instruments they may be used profitably for purposes of "general survey diagnosis," and even in some cases for class and individual diagnosis; but this work must be based more and more upon the coöperation of all concerned, from the superintendent of schools down to the pupils themselves.

In all fairness to these tests, it should be said that they have gone beyond what Professor Woody calls the "curiosity" stage and "the stage in which the predominant idea was the use of tests for determining existing levels of achievement," and, in some respects at least, have approached the third stage, "in which the predominant idea is the utilization of tests as a means for the improvement of instruction."

The purpose of this discussion is, therefore, not to condemn standardized tests utterly, but rather to point out and emphasize the need for more care in their construction, in their use, and in the interpretation of results. The matter of norms and the time element, and the place of so-called "author's norms" in published tests particularly need attention.

Misuse of Norms. When definite norms are established there is a temptation for a teacher or a school to be too well satisfied when a class or group of pupils reaches the standard norm of performance. I heard a prominent school official recently congratulating his group upon the fact that they were one above the standard norm in arithmetic. As a matter of fact, pupils of the ability of those in his school should have been considerably above the norm. Such attitudes lead to standardization of mediocrity and cause even otherwise good teachers to overlook the fact that the standard norms may be raised by lifting the general level of achievement through better methods of teaching. Someone has said, "The good is enemy of the best." The real value of a pupil's test score as pointed out by Dr. Kelley [17] in so far as a given town is concerned lies not in making comparisons with other towns, "state, or national norms, but in knowledge of differences in accomplishment found within the school system" of the particular town in question. He says further that "Ordinarily, extensive grade norms are of no importance in an educational test program, and the lack of published norms, if the test is otherwise suitable, is no hindrance to its complete serviceability in meeting the six major purposes"

[17] Kelley, T. L., *Interpretation of Educational Measurements*, p. 37. World Book Co., 1927.

which he lists for school examination programs.[18] The greatest use of standardized tests in mathematics has been made in arithmetic. This has been due to the fact that the material lends itself readily to standardization. It is only fair to say that even here the tendency to-day is away from general national standardization and toward practice exercises and diagnosis of individual cases. This has grown out of the realization of the importance of giving to the pupils their standing based upon some definite scale of performance related to their own class rather than trying to place them with reference to a norm based on the performance of some outside group—a practice due entirely to the influence of the recent testing movement.

The Time Fetish. Next comes the matter of the time element. As Dr. Thorndike has often pointed out, the speed with which a child makes errors is of no importance. We need more careful thought with reference to the purpose and place of the so-called "speed" or "time-limit" tests and "power" or "work-limit" tests. Dr. Kelley says, "Our knowledge as to the educational and social situations in which speed is of prime importance and those in which power is especially demanded is quite limited. This question is not to be settled by speculation, and relatively few experimental correlation studies comparing the merits of these two functions have been made." The tests we need to use do not presume that every pupil must do a certain amount of work in a given time; they recognize individual differences and needs. If standards are desired in a given class or school, the teacher or teachers in question should set the standard. For all these reasons standardized tests should be used more sparingly and carefully in the future.

Poorly Prepared Teachers. One thing that has impeded progress is the fact that many teachers of mathematics are not properly prepared. This is due to many causes, but two may be mentioned here as fundamental. (1) The rapid growth of secondary schools has demanded more well-trained teachers than can be supplied. (2) As a result, there has been a lack of confidence on the part of the teachers themselves as to their own fitness and advancement.

Requirements for Teachers. I heard recently of a professor in one of our prominent schools of education who told a group of prospective teachers of mathematics that the only knowledge of subject matter they needed could be obtained by taking high school

[18] Kelley, T. L., *op. cit.*, pp. 28-29.

algebra and geometry. Such advice as this is not only stupid, but dangerous. One of the most vital questions to be settled is that of the necessary qualifications for a successful teacher of mathematics. In this respect our standards are far behind those of European countries. Almost anyone can teach mathematics in the United States. We now have teachers in some schools trying to teach trigonometry in the ninth year who have never studied the subject previously. The result is obvious. It is an old story that the athletic coach is often given a class in mathematics to justify his employment in the school.

We ought to be able presently to require the calculus of all prospective mathematics teachers in the secondary schools. This subject is already required of prospective teachers of mathematics in some of our American institutions, notably at the University of Minnesota. At Teachers College, Columbia University, no one is given a diploma as "Teacher of Mathematics" or "Supervisor of Mathematics" who has not had a course in the calculus.

The matter of the importance of a teacher's personality and his chance of success needs further study. Why do teachers fail? Can we, by some scheme or other, decrease considerably the number who seem unable to succeed? Doubtless fewer teachers would fail if they knew more subject matter, but knowledge of subject matter alone will not insure success in the case of a teacher who lacks personality and the ability to understand pupils sympathetically. However, it is not always safe to conclude that a teacher is an artist merely because he obtains good results on achievement tests. Some of the best drill masters in the world have been anything but inspiring in the classroom. Teachers may be *born* rather than *made,* but surely there is something in *training*.

II. ENRICHING AND WIDENING INFLUENCES

The International Commission. This commission was appointed by the Fourth International Congress of Mathematicians in Rome in 1908 to study the teaching of mathematics in the several countries. Most of the reports were published about 1912. On account of the World War the influence of this commission was not nearly so great as it should have been, but that it was far-reaching may be seen by the preceding reports and by the reforms in our curricula in this country.

The publications of the United States Bureau of Education from

1911 to 1918 which dealt with the work of the International Commission were widely circulated. These reports showed that in regard to content, at least, we are far behind the practice in European Schools. The effect of the establishing of a new International Commission by the Bologna Congress in 1928 will be awaited with great interest.

We see by studying the work done in the other countries represented in this Yearbook that relatively unsatisfactory conditions still exist in most of our schools, although the effect of the progressive work done in Europe is having its influence upon our thinking. This has been notably true of Professors Perry [19] and Nunn,[20] and of Mr. Carson.[21]

Correlated Mathematics. Following an address by Professor E. H. Moore, before the American Mathematical Society in 1902, Professor Myers began a course in the University of Chicago High School in *correlated mathematics*. This course placed great emphasis upon the correlation and unification of mathematics in the high school. The attempt was made to eliminate the divisions between the high school mathematics subjects. He was followed by Professor Breslich, who seems to regard the work before 1916 as the experimental part of the program leading to a well-established course.

The attempt to break down the teaching of mathematics in "water-tight" compartments resulted also in an attempt to correlate the work in mathematics with that of other allied subjects, where these subjects required that mathematics be applied. This was particularly true with the physical sciences.[22] The movement in the direction of correlation seems to have become very early a matter of taking problems from other fields rather than a fusion with other subjects.[23] The tendency to correlate various subjects like mathematics and physics has thus resulted in an attempt to bring about pure correlation within the subject of mathematics itself.

[19] Perry, John, *The Teaching of Mathematics*, A Report of the British Association meeting at Glasgow in 1901. Macmillan.

[20] Nunn, T. Percy, *The Teaching of Algebra* (including Trigonometry). Longmans, Green, and Co. 1914.

[21] Carson, G. St. L., *Mathematical Education*. Ginn and Co., 1913.

[22] Myers, G. W., "A Class of Content Problems for High School Algebra," *School Science and Mathematics*, 7 : 19-33. See also note, *School Science and Mathematics*, 11 : 747.

[23] "Preliminary Report of the Committee of the Mathematics Section of the Central Association on the Unifying of Secondary Mathematics," *School Science and Mathematics*, 8 : 635-44.

General Mathematics. The movement for general mathematics of which inventional, observational, intuitive, or informal geometry is a prominent feature, dates back to John Perry's address in 1901. However, owing to the static condition of mathematics in the high school, its progress was rather discouraging until the advent of the junior high school. With the growing popularity of this institution, general mathematics in Grades 7, 8, and 9 has gained rapidly in certain quarters.

The change, however, has not been uniformly satisfactory even in the schools which go by the name of junior high schools. In mathematics, for example, we still find arithmetic occupying all the time of the pupils in the seventh and eighth grades, with a sharp line of demarcation there and then a year of traditional algebra. A corresponding situation exists in other subjects, as Professor Koos has pointed out.[24]

As early as 1917, C. B. Walsh[25] proposed a course of study having arithmetic and intuitive geometry in the seventh grade; algebra in the eighth; demonstrative geometry, partially informal, in the ninth; with elective work consisting of solid geometry, trigonometry, analytics, and the calculus in the remaining three years. It was about six years after this that the National Committee made recommendations similar in many respects. Dr. Charles W. Eliot once said, "Arithmetic, algebra, and geometry should be taught together from beginning to end, each subject illustrating and illuminating the other two."

During the period from 1900 to 1913 little seems to have been accomplished generally, but thought was turning in a direction that later took the form of general mathematics. Historical and psychological evidence were offered to justify the rearrangement of materials of instruction.[26]

The era of scientific attempts to combine the topics of mathematics for teaching purposes was preceded by experimentation with a plan of alternating the teaching of algebra and geometry. The

[24] Koos, L. V., *The Junior High School* (enlarged edition), pp. 168-76 and 244-45. Ginn and Company, 1928.

[25] Walsh, C. B., "A Tentative Program of Junior High School Mathematics," *The Mathematics Teacher,* 10 : 85-93.

[26] Bass, Willard S., "The Historical Argument for Teaching Arithmetic, Geometry and Algebra Together in the First Year of the High School," *School Science and Mathematics,* 5 : 712-16. See also 6 : 495-500 ; also a report, 8 : 70-74, and Myers, G. W., "Two Years' Progress in Mathematics in the University High School," 11 : 64-72.

first was taught for two or three days of the week and the remainder of the time was devoted to the latter.[27]

The vicious attacks on the "water-tight" compartments has been much more effective than seems apparent at first sight. Most of the currently published mathematics texts call themselves algebras or geometries and thereby seem to retain the "water-tight" system. However, a careful examination of the content will show many places where algebra has found a place in the work in geometry or the reverse. Some authors indicate a half-way attitude by their titles "Modern Algebra," "Modern Geometry," "Essentials of Algebra," or "The New Geometry."

The National Committee on Mathematical Requirements. This committee was appointed by the Mathematical Association of America in 1916 and was financially assisted by the General Education Board. The report[28] of the committee, *The Reorganization of Mathematics in Secondary Education,* was published in 1923. Professor Smith has summarized this important work as follows:

> This report was prepared in close coöperation with bodies of teachers throughout the country. It set forth very clearly the aims of mathematical instruction in the several years of the junior high school, the senior high school, and the older type of four-year high school. It presented the model courses for these several types of school and made suggestions for carrying out the work. It considered the question of college entrance requirements, the basal propositions of geometry, the rôle of the function concept, and the terms and symbols which might properly have place in the schools. It fostered various other investigations, including the present status of the theory of disciplinary values, the theory of correlation applied to school grades, a comparison of our curricula with those in use abroad, experimental courses in mathematics, standardized tests, and the training of teachers. It is not too much to say that the advance in the last decade has been due in large part to the work of this committee.

These recommendations have had a very great effect upon the curriculum in both junior and senior high schools, but they have been more generally accepted in the junior high school. The report was being formulated at the time the course of study for the junior high school was being planned and for this reason probably influenced the curriculum at the most plastic time in its history. The

[27] Myers, G. W., *loc. cit.* See also "Provisional Report of the National Committee of Fifteen on Geometry Syllabus," *School Science and Mathematics,* 11 : 509-31 ; and Myers, G. W., "Report on Unification of Mathematics in the University High School," *School Science and Mathematics,* 11 : 767-90.

[28] The original report is out of print, but a revised edition containing most of the important features can now be obtained from Houghton Mifflin Co., New York.

first junior high school textbooks appeared about 1917 at the time that the committee was at work on its report.

The advertisements of publishers of certain modern textbooks, stating that such and such a book follows the recommendations of the National Committee, even though referring to a plane geometry text, show the real thoughts of teachers in the field and that the committee's report is beginning to be felt.

The Junior High School. The junior high school movement began to be an important factor in American education about 1915. This movement created a situation in the seventh, eighth, and ninth grades which was largely independent of the school situation as these grades were commonly organized. Here was a chance for progress, and we find a considerable change made in the curriculum in many schools. Many textbook writers on junior high school mathematics organized the work in units and introduced considerable intuitive geometry and some algebra in the seventh and eighth grades. Most of them have introduced trigonometry in the ninth grade, and some a unit of demonstrative geometry. This junior high school course at its best is a revealing and exploring experience for the children, opening up large vistas in materials and situations which are helpful and meaningful. Thus we see that the movement for general mathematics, while it has not been accepted in most of the four-year high schools, has been adopted by many junior high schools and made a part of their program.

In the junior high school the course in mathematics has been better organized than that of any other subject, but it has been principally a shoving down of the traditional material from the senior high school field. One needs only to examine a few of the many series of mathematics texts to see that this is true. As a result the senior high school teachers generally object to the teaching of certain topics, like demonstrative geometry, in the junior high school because they feel that such teaching will unfit the pupils for later work in the same subject in the senior high school.

Effect upon the Senior High School. This junior high school course of study is having its effect upon the senior high school. Certainly, if such a course is good for the children in the ninth grade in a junior high school, it should be good for the children in the ninth grade in a four-year high school. The tendency now in the tenth grade is to modify the idea that every proposition in geometry must be proved rigorously. Teachers now accept more

facts intuitively and build upon these. There is less tendency in the tenth year to offer a general course, although there is a trend toward this in some schools.

Effect upon the Elementary School. The new course of study for the junior high school is also having its effect upon the seventh and eighth grades in the eight-year elementary school. There is a tendency to introduce more intuitive geometry and perhaps a little more algebra into these two grades. We also find more work in statistical graphs. For the most part, the work is quite different from that in the average junior high school, as far as the mathematics curriculum is concerned.

Experimental Schools. The growth of experimental schools and the frequent reports in the *Mathematics Teacher,* in the National Council *Yearbooks,* and elsewhere, of experimental work that is being carried on by teachers in public schools indicate the great interest in better presentation, better teaching, and better learning of mathematical material whether old or new. Experiments with individual instruction, homogeneous grouping, laboratory instruction, large versus small classes, and the like, indicate a professional interest on the part of teachers everywhere that is stimulating and suggestive of an evolving trend.

The mathematics department of the University High School at Chicago, that of the Horace Mann School, and that of the Lincoln School of Teachers College, Columbia University, for example, have shown conclusively that it is possible to give their pupils an interesting and modern course in mathematics and at the same time prepare them to pass college entrance examinations.

Contribution of Psychology. The development of psychology from a speculative philosophy to an empirical science has affected both the content material and methods of instruction in our schools. The pupil, his capacities, and his needs have come in for an amount of attention never before accorded to them. In other words we are attempting to get the pupil's point of view.[29] Although the greatest contribution has been made in the elementary field, the influence of advanced thinkers like Professor E. L. Thorndike has been felt all along the line. To be sure, the psychologists have shown us how to teach better some things which would better go untaught, but they have also helped us to organize our fundamental material along

[29] Smith, David Eugene, "Teaching of Mathematics in the Secondary Schools of the United States," *School Science and Mathematics,* 9 : 209. See also 9 : 90.

lines that are psychological rather than logical. Thus, treatment of subject matter has been made more concrete, content material has been organized in terms of the learner instead of the subject, and the entire atmosphere of the learning situation has been improved. Drill, which was once looked upon as drudgery, can now be introduced as a sort of game in which a pupil's competition against his own previous score or record may become the motive for self-improvement. The psychologists have discovered many useful facts and laws about how children learn most easily and most economically, how habits are formed, how abilities are developed and how they may be retained.

Influence of the Educational Philosophers. A few philosophical writers like Professor Dewey [30] have had great influence on classroom teaching in America. They have recognized the American child, have emphasized his possibilities and his rights, and have encouraged the modification of the course of study with these things in mind.

Due in part to the emphasis placed by Dewey and others on the importance of interest and purposeful activity in connection with school work, efforts have been made to "vitalize and motivate" the work in mathematics. The project method has been used considerably, especially in the lower grades. It has not found favor, however, in the secondary schools; and its use in elementary schools is condemned by some authorities. Other thinkers also interested in the welfare of children nevertheless remind us that subject matter itself has certain rights and that the possibility of a more careful consideration of content material in curriculum construction must not be overlooked.

Influence of the Educational Sociologists. The educational sociologist has also influenced the curriculum, bringing the social needs of the pupil more prominently before the teachers. As a result, the tendency has been not only to place the pupil and his development at the center of educational interest but also to modify and reorganize all subject matter on the basis of its learning difficulty.

Social Utility as a Basis for Curriculum Construction. The recent syllabus [31] of the State Department of Education in New

[30] Dewey, John, *Interest and Effort in Education.* Houghton Mifflin, 1913.

[31] *A Tentative Syllabus in Junior High School Mathematics.* The University of the State of New York, 1927.

York states, "Particular attention should be directed to the danger of emphasizing 'social utility' as the sole basis of the curriculum." The syllabus then quotes this passage from the *Twenty-sixth Yearbook* of the National Society for the Study of Education:

It is exactly that tendency of individual human judgment to lose its bearings and fail to see the woods for the trees, that has led the more scientifically minded students of education to take the basis of curriculum making out of the realm of individual judgment. They have been experimenting of late with the criterion of social utility and especially with objective bases of selection. It was natural in the first rush of the movement, with the initial impulse to play with the new idea, that its disciples should be carried to extremes. It cannot be doubted that many of our workers to-day are dominated by the belief that only those facts, principles, and motives shall be taught in the school which can be utilized immediately and generally by a considerable proportion of our people. If perpetuated, this attitude will result in a mechanistic curriculum of the rankest sort. This view is already serving to make uncritical workers overemphasize the skills and the factual knowledge of the curriculum.

Another quotation from the same *Yearbook* will bring this point into relief. It reads:

Now, it is of great importance for the curriculum maker to see that the determination of goals for a given social order will be most soundly made when he has at hand adequate knowledge and a deep and broad perspective of that social order. The task of stating the goals of education, therefore, is not to be consummated by an analysis of social activities alone. It will be aided by the latter, but must not be dominated by it. It will be achieved only by hard thinking and by the most prolonged consideration of facts by the deepest seers of human life. For the great bulk of our curriculum, therefore, the analysis of social activities will influence the judgment of the seer based upon the scientific study of society—not the mere factual results of social analysis—that will determine the more intangible, but directing materials of our curriculum.

Social analysis merely gives us the techniques and knowledges we should have. For the basic insights and attitudes we must rely, as we do for the statements of the goals of education, upon human judgment. It is imperative, however, that we make use of only the most valid judgments. The forecasting of trends of social movement, the perception of the local problems and issues, and the connections underlying them, demand erudition and maturity of reflection that eventuates only from prolonged and scientific study of society. To the frontier of creative thought and of deepest feelings we go for guidance as to what to teach.

Widening Influences. We see a widening influence, first in mathematics in the gradual dropping down into the lower grades of

algebra, trigonometry, intuitive and demonstrative geometry, spherical trigonometry, and finally calculus. None of these is taught as a complete topic, but each as an element of more vital and significant material. The old material which it replaced was taught according to the theory that more of a topic was needed to prepare for more of the same topic and all for its own sake.

In the second place, we find a widening of the whole field to include pertinent and significant parts of other fields. The *Third Yearbook* [32] of the National Council indicates the present reaching out into the fields of physical measurements, surveying, use of the slide rule, and the like. This, after all, is merely a continuation of the progressive attempt to put life into the formal subject matter begun three decades ago. The introduction of the elements of the calculus will facilitate the forming of closer bonds with the sciences, since a complete understanding of the calculus in elementary fields must come in its big field of applications. In the past we have criticized the college texts as narrowing, pedantic influences, but in such books as Griffin's *Introduction to Mathematical Analysis*, Lennes's *Survey Course in Mathematics*, and Mullins and Smith's *Freshman Mathematics*, there is this widening in two directions.

III. ORGANIZING INFLUENCES

Four Steps in Curriculum Building. Thus far we have considered some of the general aims of American education and some of the large outlines of our main problem, but they by no means equip us with the details for classroom procedure. The educational problems of most interest at the present time relate to the curriculum. They extend from the National Education Association through its various divisions down to the smallest educational unit. This is evidenced by the various yearbooks that have recently appeared from such organizations as the Department of Superintendence of the N.E.A.[33] the National Society for the Study of Education,[34] and the National Council of Teachers of Mathematics.

In determining a curriculum suitable for our schools we are concerned with four large features; namely, the objectives to be

[32] The *Third Yearbook of the National Council of Teachers of Mathematics.* Bureau of Publications, Teachers College, Columbia University, 1927.

[33] *Third, Fourth, Fifth,* and *Sixth Yearbooks* of the National Education Association, Department of Superintendence.

[34] *First* to *Twenty-seventh Yearbooks* of the National Society for the Study of Education.

attained, the content material best suited to accomplish these objectives, the best methods of teaching and learning, and a testing program. We shall discuss each of these briefly here and later give a more complete treatment of the details of each.

Speaking of the traditional subjects of the curriculum Professor Briggs [35] says:

> The traditional subjects of the curriculum have for some time been vigorously attacked by those dissatisfied with results and skeptical of improvement. Curriculum makers are proposing new subjects and new phases of subjects, which still further put on the defensive the traditional program. I very much doubt if we are going to depart materially for some years to come from the science, the mathematics, the languages, the social studies that have been the pabulum of generations. But if they wish to survive, they must adopt the basic doctrine of interest. This will necessitate a new attitude on the part of teachers and constantly new plans of organization rather than a revolution in content.

1. *Aims.* In the first place we cannot expect to realize our aims unless they are precise and are clearly defined at the outset. These aims should not be imposed upon the schools by some higher authority; on the contrary, they should be the result of much discussion on the part of those who will be actually responsible for their realization, who know the work of the classroom, and who themselves are the authority as to what can be expected of children. It is evident that no satisfactory list of objectives can be set up merely by consulting existing courses of study. While they may be used as evidence of what *is* being taught throughout the country, they do not often give much help in suggesting what *ought* to be taught. Moreover, it is well known that some of these courses, if not all, tend to perpetuate certain obsolete processes and antiquated business methods.

It is equally true that the best objectives cannot be secured by making an inventory of the current textbooks in mathematics. In most schools they are the courses of study. They, too, are frequently guilty of overemphasizing unimportant or the obsolete material. It is also true that not all textbook writers are able to suggest newer and better things. Often their books are made to conform to certain state syllabi or other courses of study.

The standardized-test makers of recent years have erred in

[35] Briggs, T. H., "Interests as Liberal Education," *Teachers College Record,* 30 : 674.

including exercises and problems that thoughtful teachers everywhere have no desire to see perpetuated in our schools. In fact, many of these undesirable elements were obtained by the makers of tests from existing courses of study and textbooks. Thus, it is obvious that such tests cannot be used as the sole basis for determining a list of desirable objectives.

We also know that it is not safe to try to determine what mathematics ought to be taught by counting the frequency with which certain mathematical terms are used in a few current editions of newspapers and magazines, although some writers seem to believe that such procedure is valid.

Finally, it is fair to say that we cannot determine our objectives by going out in the world and asking different individuals what mathematics is useful to them. The fact is that not one of them ever knows just what use he has made of mathematics. Moreover, they probably have given no thought to the question of determining how they might have used mathematics profitably if they had known more about it.

Any and all of the above criteria may be of service to us in making up a list of desirable objectives, but they will not suffice. If the objectives set up are to meet modern needs, we must have at least one other criterion that is supported by vision, though it should not be visionary. This last criterion is the opinion of expert teachers of mathematics—those who are able not only to tell how mathematics is used, but also to show how it may be used in the present and in the future for the betterment of mankind.

People consult with experts in order to get the opinion of the best they can afford or find. Such a group of experts recently cooperated in setting up a list of objectives [36] in mathematics for the junior high school, but the list is too long to include here. Professor Bode said [37] recently:

> Education must provide an escape from the bondage of tradition by fostering a realizing sense that human history is the record of a great adventure in which man continually recreates his social and moral standards. On this level his criteria of values become mundane and experimental, not transcendental and final, and man's aspirations and efforts are directed more and more to the realization of a new and growing social ideal.

[36] Smith, David Eugene, and Reeve, W. D., *The Teaching of Junior High School Mathematics*, Chap. III. Ginn and Co., 1926.

[37] Bode, B. H., "The Most Outstanding Next Steps for Curriculum Makers in the United States," *Teachers College Record*, 30 : 187.

In setting up objectives we must go beyond the mere device of the scientist who collects and interprets his data, especially when "the situation calls for a recreating or reinterpreting of old ideals." To quote Professor Bode [38] again:

> The problem of ultimate objectives is not a scientific problem at all. The function of science is to reveal what agencies can or might be employed to realize ultimate objectives; it does not undertake to determine the objectives themselves.

The implication is that we must depend upon a broad and inclusive philosophy of education, the business of which is to find out what ought to be the best education in mathematics for every person in the community.

2. *Content.* Having selected a valid list of objectives to which all concerned agree, we are confronted next with the task of selecting that content material which will best enable us to realize these objectives. This will be elaborated later.

3. *Methods.* The third step in curriculum building relates to method. Usually the use of the word "method" suggests "method of teaching," but as we view the situation to-day we usually consider another side; namely, "method of learning" most easily and most economically.

We know a great deal about the former, but practically nothing about the latter. To-day we do not know how long it takes to teach anything. Granted that the objectives chosen are valid, that the content material selected is best suited to realize these aims, and that our methods of teaching are satisfactory, we still need to find out what part of the content material *can be learned* by a pupil at a given age. It is probable that in some cases we are trying to teach too much, or that we are attempting to teach things that are too difficult, or both.

At the beginning of the Twentieth Century, classroom procedure was well standardized. The teacher assigned the lesson, sometimes with little or no attention to the development in class of the main points, and frequently by assigning a certain number of pages. The pupil then went about the learning of the lesson as best he could and the next day recited what he had learned. This resulted in memoriter methods where often reasoning was desired, especially in a subject like geometry. In a subject like algebra such teaching

[38] *Ibid.*, p. 190.

resulted in the development of a facility in dealing with formal symbols and abstract ideas without ability to apply them, and all too often with no tangible appreciation of their real significance.[39]

4. *The Testing Program.* Suppose that we are able to set up a list of desirable objectives, that we have mastered the best known methods of teaching, and that we are well informed in the psychology of learning, how can we expect our work to be successful without an elaborate testing program? Some of the objectives which we set up may be too difficult for a child at a given age to attain; others may be too easy. We know now that the mastery of most of our topics in some of the fundamentals is pitiful. We should have known that a long time ago, had we tested more. Suffice it to say that the testing movement is now in need of serious study. The emphasis at the present time needs to be put upon the use of tests for the improvement of instruction. The present abuses connected with the use of norms and standards need to be corrected, and it is our duty to make the testing a vital and integral part of the entire program of curriculum construction.

Problems in Teaching Mathematics. The problems that concern us here relate to the pupil, to the teacher, and to the materials of instruction. We shall take up each of these in our subsequent discussion.

Aims of Instruction. The term "aims of instruction" takes its literal significance and hence needs no further explanation. It is now generally recognized that the big problems in American education "are problems of aim." No one will deny that these aims have changed greatly in the last thirty years. We shall see why this has been so. No efforts to discuss aims or methods were actually made in the N. C. A. until about 1903 when Professor Moore's committee made its report. This was one of the influences that helped to develop our modern practice of stating objectives.

The Secondary School. The secondary school grew up in the United States naturally with the curriculum so organized as to meet the needs of those pupils who intended later to go to college or at least to follow intellectual pursuits. This is why the college professors determined pretty largely the subject matter to be presented in the schools. Because of the enormous increase in attendance in the high school and the wider interest in public education,

[39] Perry, John, "The Preliminary Education of the Engineer," *School Science and Mathematics*, 2 : 264-72.

attention has been given recently to the needs of those students who do not intend to carry their formal education beyond the secondary school. Thus, the senior high school has become the "people's college." Naturally, this has resulted in a considerable modification of our traditional courses of study,[40] and has presented us with many problems difficult to solve.

Where the aim was formerly to develop scholars, the aim now is to develop well-educated citizens. In many schools to-day we are developing neither, principally because we are making a fetish of education. We believe that this is justifiable for social reasons and for the safety of the nation.

Modern Aims. By the year 1910 psychologists for the most part had rejected the faculty idea of psychology, and mathematicians, as well as other educators, were looking for the best way out of the dilemma. We find two different views of the value of mathematics emphasized by those interested in the subject. They may be stated briefly as follows: (1) To emphasize the cultural value of mathematics and to regard it as fundamental in the training of the power to think clearly and logically. (2) To supply pupils with information which they can use in the ordinary pursuits of daily life. The trend now is back to an emphasis upon the transfer value of training. Such leaders as Professors Judd, Nunn, and Hedrick are emphasizing the fact that mathematics has considerable value other than for the information it imparts.

Increase in School Population. According to Professor Thorndike,[41] not only has the increase in the high school population been so great that to-day "almost one in three of the children reaching their teens in the United States enters high school" against the corresponding figures of one in ten in 1890, but the pupils of to-day "are different from those of twenty-five years ago, not only in their experiences and interests, but also in their inborn abilities." Moreover, "the number of high school pupils in 1918 was six times that in 1890, while the number of children of high school age in 1918 was less than one and two-thirds times that in 1890. The number graduated, which is in some respects a better measure, was eight times as large in 1918 as in 1890."

Professor Thorndike further points out that "We lack measures

[40] "Missouri Society of Teachers of Mathematics and Science," *School Science and Mathematics*, 8 : 300.

[41] Thorndike, E. L., *The Psychology of Algebra*, p. 2. Macmillan, 1923.

of the inborn capacities of the one in ten or eleven of a generation ago and have only very scanty measures of the capacities of the one in three to-day. We have, however, excellent reasons for believing that the one in ten had greater capacities for algebra and for intellectual tasks generally than the one in three of to-day." Such a situation presents other problems that are difficult to solve.

In spite of all that has been said above, the facts show that "the pupils in our academic high schools are, in fact, a limited group which covers just about half, the upper half, of the total distribution of American intelligence." Why, then, the enormous number of failures that we find over the country in the first year of the ordinary four-year high school? These failures constitute a problem the solution of which is going to challenge the best thinking of the leaders in education.

The change in objectives that has come about in the last decade may be expressed by quoting the aim of mathematics formulated by the National Committee on Mathematical Requirements as follows:

> The primary purpose of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and of space which are necessary to a better appreciation of the progress of civilization and a better understanding of life and of the universe about us, and to develop those habits of thinking which will make these powers effective in the life of the individual.

Materials of Instruction. This term refers to topics and sources generally that are usually drawn upon to furnish the subject matter for the course in mathematics.

The problems relating to content are so numerous and complicated that we can take time to discuss only a few. First, there is the question as to whether the traditional course in arithmetic, followed by algebra in the ninth grade, geometry in the tenth, and so on, is the best one to perpetuate. Perhaps we shall not want to perpetuate any plan. The widespread failures already referred to are evidence enough to show that the traditional organization is not satisfactory.

There are many who feel that a general mathematics course beginning in the seventh year and extending through the high school with the calculus as the last objective would be the best way of organizing the subject matter. The course in the regularly organized

junior high school is a general one, but this is not true in most seventh and eighth grades not organized on the junior high school plan. This is certainly most unfortunate. What is good for the pupil in one school in the seventh grade should be good in another. Moreover, we are not agreed upon some of the main issues. For example, we are not agreed as to whether the course should begin with arithmetic, which in many cases is a review, or with intuitive geometry. Some of us think that it would be psychologically more sound to begin with the latter. Certainly we shall err if our seventh grade work in arithmetic consists mostly of drills on the fundamental skills taught in the earlier grades.

Attempts have been made to set up an acceptable list of objectives [42] which represent what is actually being done in junior high schools. We should remember, however, that merely setting up a list is not sufficient. Such objectives should be more widely discussed and understood.

In building a curriculum we have had to consider whether we should have a unit of demonstrative geometry in the ninth year, whether algebra would not be better understood if it were begun in the seventh year and then scattered through the eighth and ninth years, and other similar problems.

Progress in the senior high school has been slower. While teachers in the elementary school may still question whether the course in arithmetic is entirely satisfactory, they have improved the course greatly. We have seen how the teachers in the junior high school have built up a progressive course.[43] On the other hand, the teachers in the senior high school have been more or less content to let things stand as they are. The course in algebra has been improved in content, numerical trigonometry has been taught in some ninth grades, but generally speaking the situation is static.

Changes in the Course of Study. Traditionally materials have been organized in "logical" units—putting together subject matter which logically belongs together. Then in studying the material, the pupil concentrates intensely on one topic at a time without giving thought to its relation to other topics or to the course as a whole. To attain lasting results to-day, it must be clearly under-

[42] Schorling, Raleigh, *A Tentative List of Objectives in Junior High School Mathematics*. George Wahr, Ann Arbor, Michigan, 1926.

[43] Smith, David Eugene, and Reeve, W. D., *The Teaching of Junior High School Mathematics*, Chap. III. Ginn and Co., 1927.

stood that the organization must be made in "pedagogical" units, rather than "logical" units. This organization of mathematics is best exemplified in the new mathematics course of the junior high school. Giving the pupil a wide experience with the fundamentals of mathematics makes possible a gradual and easy approach to the parts of the subject taught in the senior high school, whether or not the material there is a general course.

Omissions and Additions. With the elimination of obsolete material inserted when it was considered that everything mathematical was useful, has come the bringing down of the most useful and interesting concepts from higher mathematics into the junior and senior high schools.

Partly as a consequence of the greater importance attached to the child and his interests, and partly as a natural growth of professional interest in the subject itself and a desire to exhibit its maximum value, there has been a very decided tendency to eliminate much that was commonly found in our textbooks in 1910 and substitute more valuable material.

In place of these dreaded topics we have substituted work in informal geometry, meaningful formulas, graphs, numerical trigonometry in the ninth grade, and so on. Early introduction of coördinate geometry has made possible the introduction of the calculus in the eleventh or twelfth year in many schools. The idea of functional relationship between variables is creeping into all texts more or less conspicuously as the unifying element. We shall now discuss some of these omissions in more detail.

Modified View of the Theory of Mental Discipline. With our modified view we do not look upon all topics in a given subject as having equal value. We no longer require pupils to study mathematics because they find it hard or because they dislike it, but because we can find parts of the subject essential to the education of every American citizen. The elimination of certain complicated, difficult, and unreasonable exercises and problems can be traced immediately to the fact that nothing but a mystical disciplinary value could possibly justify their retention. This is seen in the special fields of arithmetic, algebra, and geometry where the trends have been peculiar to these fields.

Arithmetic. The arithmetic of special and unusual occupations has been replaced by the arithmetic of the daily life of the people; obsolete methods and processes have been eliminated. In place of

giving all topics equal importance those obviously most essential to life problems are stressed.[44]

In the following table some types of material that were set forth in one arithmetic approximately ten years ago are contrasted with the modern offering:

THEN	NOW
Arithmetic of special and unusual occupations. For example, problems involving	Arithmetic for daily life. For example, problems involving
1. Partial payments.	1. The home.
2. Marine insurance.	2. Banking.
3. Measurements of hogsheads.	3. Daily purchases.
4. Long and unusual fractions.	4. The farm.
5. Tax collector's commissions.	5. Fractions used in everyday life.
6. Hard additions and subtractions.	6. Business.
7. Partnership including the question of time.	7. Industry.

Simplification of Algebra. Among the first topics to be eliminated from elementary algebra were the highest common factor by division, cube root by the formula, the general theory of the quadratic, complicated brackets, complex fractions of a difficult type, simultaneous equations in more than three unknowns, the binomial theorem, and complicated radicals. The old idea that "we must scientifically define all terms before they can safely be used" and develop the subject logically has been replaced by a psychological development. One might here raise the question why the study of quadratic equations beyond the pure type such as $x^2 = 4$ should any longer be required of everybody in the ninth grade. Those who continue the study of mathematics will have to consider the topic anyway and those who do not will never have any use for the kind of work that is traditionally given. It is artificial and ought to be omitted.

Graphs. The graph, of great and growing importance, began to receive the attention of mathematics teachers during the first decade of the present century.[45]

Teachers are seeing that the best approach to algebra is not made by means of the equation or through the fundamental opera-

[44] See Smith, David Eugene, *The Progress of Arithmetic.* Ginn and Co., 1923. See also Thorndike, E. L., *The Psychology of Arithmetic,* Macmillan, 1922; and *The New Methods in Arithmetic,* Rand McNally, 1921.

[45] Palmer, Emily G., "History of the Graph in Elementary Algebra in the United States, *School Science and Mathematics,* 12 : 692-93.

tions but through the study of the formula. The study of the graph is a major trend to-day in algebra because with the formula it helps to clarify the idea of functionality. We now emphasize the meaning of graphs rather than the making of them.[46]

The graph appeared somewhat prior to 1908 and, although used to excess for a time, has held its position about as long and as successfully as any proposed reform. Owing to the prominence of the statistical graph, and the increased interest in educational statistics, graphic work is assured a permanent place in our courses in mathematics.

Functional Thinking. A significant trend in the teaching of algebra, not fully realized but well under way, is to put meaning into the subject by replacing the emphasis upon formal symbolism by the function concept. The slowness with which this idea has been adopted in teaching probably accounts for the despair with which an occasional educator regards algebra.[47] The more progressive courses are now planned so as to bring out the dependence of variable quantities on each other at every possible opportunity. In some schools trigonometry, analytic geometry, and the calculus are introduced at this point to help bring this about. Professor Nunn says:[48]

> As soon as the symbols of trigonometrical ratios are recognized as capable of entering into formulas and of being manipulated, they should be regarded as belonging to the vocabulary of algebra. There is, indeed, no principle except the invalid principle of formal segregation, upon which we can include the study of x or a in the algebra course and exclude $\sin x$ or $\tan x$. All alike are pieces of symbolism invented for the description and interpretation of facts of the external world. Each represents a typical kind of function. To each corresponds a specific form of curve which may be regarded as the graphic symbol of the function. Both algebra and trigonometry would gain by fusion—the former through an added variety and richness in the illustration of its main themes, the latter by the removal of the excessive formalism which at present obscures its value and interest for the beginner.

Old and New Syllabi. Additions and omissions may be best illustrated by reference to the New York State Syllabus of 1910, and to the new syllabus which will go into effect next year:

[46] Hedrick, E. R., "The Reality of Mathematical Processes," *Third Yearbook of the National Council of Teachers of Mathematics*. Bureau of Publications, Teachers College, Columbia University, 1928.

[47] Hedrick, E. R., "On the Selection of Topics for Elementary Algebra," *School Science and Mathematics*, 11 : 51-60.

[48] Nunn, T. Percy, *loc. cit.*, pp. 19-20.

SYLLABUS OF 1910	NEW SYLLABUS
1. The ability to formulate necessary definitions in clear concise language.	1. No mention of definitions as such, but algebraic language and representation substituted instead.
2. Removal of symbols of aggregation, and insertion of terms within such symbols.	2. Removal of one set *and at the most two sets* of symbols of aggregation.
3. Factoring of expressions of four terms and of those with literal exponents.	3. Taking out a common monomial factor and factoring the difference of two squares. Factoring trinomials optional.
4. Application of the principles of factoring in finding H.C.F. and L.C.M.	4. The terms H.C.F. and L.C.M. as such not taught.
5. Fractions including complex fractions of the "apartment house" type.	5. Fractions no harder than those needed in the most difficult formula taught.
6. Ratio taught as a separate topic and all definitions given.	6. Ratio treated as a fraction.
7. Proportion: Inversion, alternation, composition, and division were taught.	7. Proportion treated as a fractional equation.
8. Radicals: Definitions. Rationalization when denominator is a binomial surd. Radical equations. No mention of fractional exponents.	8. From the beginning pupils are to be familiar with the use of the fractional exponent.
9. Quadratic equations: Solution of pure quadratics and complete quadratics, by factoring, by completing the square, and by formula.	9. Study of quadratics optional. Tendency to teach only the formula method of solution.
10. Radical equations resulting in quadratics.	10. Radical equations omitted.
11. Binomial Theorem for positive integral exponents.	11. Binomial theorem omitted.
12. Graphs omitted.	12. Graphs. Simple statistical graphs. Representation of the formula by the graph. Interpretation of the graph.
13. Graphic representation of directed numbers omitted.	13. Graphic representation of directed numbers.
14. Function concept omitted.	14. Function concept.
15. Numerical trigonometry omitted.	15. Numerical trigonometry.

Algebra. Arising very largely as a reaction to the overemphasis on manipulative skills in mathematics, attention has been

called to the importance of teaching the pupil not merely to obtain the correct answer but to think about and understand the meaning of the operations he performs. In a recent study Professor Everett [49] points out the need for a more careful analysis of the objectives in algebra and a more intelligent teaching program.

Geometry. Geometry, the oldest and most logical structure, has naturally resisted change more than any other part of mathematics, and the changes are mostly on the surface. The texts are better, the amount of memory work has been diminished, and there has been more emphasis on original work. The fundamental change has been the introduction of the intuitive geometry in the seventh and eighth grades, and a short unit of demonstrative geometry in the ninth grade. This preliminary work has been one of the main factors in reducing the time spent on the subject, so that one year of plane and solid geometry combined in the tenth grade is now thought by some authorities to be sufficient.[50]

Fusion of Plane and Solid Geometry. As early as 1905 there was apparent not only the idea of a greater emphasis upon intuitive geometry and "original exercises," but also upon the fusion of plane and solid geometry. At least there seems to be no very good reason why the combination of these two parts of geometry into a one-year course for the tenth grade should not be made, and the present tendency is in this direction. No one can teach all of geometry in a lifetime anyway and the important part of solid geometry for the well-educated citizen should not require a half-year of study. Moreover, why should we live in a world of *three dimensions* and teach the geometry of *Flatland?*

Intuitive Geometry. When the doctrine of formal discipline came into question certain adherents of the doctrine began to assume a defensive attitude. Some people tried to justify the retention of traditional geometry in the curriculum by calling attention to its applications, quite overlooking the fact that the science of Euclid's day as universally taught had apparently developed in the direction of pure logic. The unmarked straight edge and the compasses, the instruments to which the pupil is limited in studying geometry, make it impossible to give practical applications of pure Euclidean geometry to the physical world.

[49] Everett, J. P., *The Fundamental Skills of Algebra.* Bureau of Publications, Teachers College, Columbia University, 1928.

[50] Allen, Gertrude, "A Modified Program for School Geometry," *University High School Journal,* 4 : 269-78. Oakland, Cal.

In order to dissipate the confusion into which the subject and those who taught it had been thrown by the overzealous and logical geometricians, a movement was started which resulted in the separation of geometry into what are now known as intuitive and demonstrative geometry. The term "informal" would doubtless be better than "intuitive" because the former might then be used in connection with the so-called "informal proofs" (in contrast to "formal proofs") which would include proofs by "intuition" and by "experiment." Thus, "informal" would be the broader term. The trend in this direction, embracing a practical effort toward making the introduction to demonstrative geometry more gradual and natural can be discerned early in the present century.[51] It is interesting to note that at that time the adjectives "inventional," "concrete," and "observational" were commonly used to convey the ideas now expressed by the word "intuitive" or "intuitional."

It is probable that the development of this idea of a definite course in informal geometry of the kind indicated above will stand as one of the notable advances of the last quarter of a century in the teaching of elementary mathematics.

Postulates. Closely paralleling the trend in the simplification of arithmetic and algebra by omissions and reorganization of materials came the suggestion that in geometry theorems whose meaning was already perfectly clear and obvious to the pupil should be postulated. The feeling is growing in our schools that the rigorous demonstration of such theorems should either be omitted entirely, or be deferred until the pupil's knowledge of geometry has advanced to a point where the logical implications of a proof have some significance for him. Teachers are beginning to see the conflict between the adult logic of pure science and the laws of learning by which their pupils are governed. In such cases learning for the pupil ceases to be a burden.

Limits. Within the last few years the theory of limits has been omitted in the elementary courses. Even though the topic is entirely omitted from most of the recent textbooks, it was commonly taught twenty years ago in most schools. In spite of very strenuous opposition, it occupies to-day a subordinate position in modern courses of study.[52]

[51] Hart, Clara A., "The Teaching of Geometry," *School Science and Mathematics*, 5 : 717-25.

[52] Lennes, N. J., "The Treatment of Limits in Elementary Geometry," *School Science and Mathematics*, 6 : 52-58. See also Betz., Wm., "The Teaching of Geom-

Applied Problems. There is at the present time an apparent desire to introduce a reasonable number of applied problems rather than to depend upon abstract propositions alone, and to arouse the interest of the pupil by an appeal to situations within his comprehension. It is even asserted by some that culture itself can be practical and taught in conjunction with things that are practical. In other words, it is not believed to-day that mathematics must be taught altogether as a pure science.[53]

Numerical Trigonometry. The introduction of simple numerical trigonometry in connection with the work in ninth-grade algebra is one of the important additions of the last few years. Professor Smith says that it is the most notable step forward in the last quarter of a century in the progress of algebra.

Calculus in the High School. A suggestion of the possibility of teaching calculus in the high school was offered as early as 1910,[54] but the idea had not then been given the authority of actual trial. At intervals since 1910 there has been a great deal of agitation, especially noticeable at present, for teaching the fundamental elements of differential and integral calculus. Moreover, in not a few high schools in this country, the more progressive teachers have been experimenting in various ways trying to improve the traditional organization. Such work is done in the Horace Mann School and the Lincoln School of Teachers College, Columbia University; in the University High Schools at Minneapolis and at Oakland, California.[55] Mr. John Swenson[56] is doing a notable piece of work in teaching the calculus to young girls in Wadleigh High School, New York City. The reader should consult other articles on the senior high school program in mathematics for a more complete and suggestive outline of proposed work in grades ten, eleven and twelve.[57]

etry in Its Relation to the Present Educational Trend," 8 : 625-33 ; and Lytle, Ernest B., "Limits in Elementary Geometry," 10 : 530-32.

[53] "Report of a Committee on Real and Applied Problems in Algebra and Geometry," *School Science and Mathematics*, 9 :788-98.

[54] Smith, David Eugene, "Teaching of Mathematics in the Secondary Schools of the United States," *School Science and Mathematics*, 9 :203-19. See also Collins, Jos. V., "The Perry Idea in the Mathematics Curriculum," *School Science and Mathematics*, 12 : 296.

[55] Durst, Ethel H., "Calculus for High School," *University High School Journal*, Oakland, California, 6 :135-67.

[56] Swenson, John, "Selected Topics in Calculus for the High School." *Third Yearbook of the National Council of Teachers of Mathematics*. Bureau of Publications, Teachers College, Columbia University, 1928, pages 102-34. See also Nordgaard, M. A., "Introductory Calculus as a High School Subject," pages 65-101.

[57] Reeve, W. D., "The Mathematics of the Senior High School," *Teachers College Record*, 27 :374-86. See also Mirick, G. R. and Sanford, Vera, "An Elective Course

Elementary Statistics. The prospects are that we shall soon give more attention in the mathematics program to elementary statistics. A few schools are already doing this, although the textbooks [58] generally do not treat it. The demand in several fields like education and economics makes a knowledge of the most elementary notions of the statistics of variables a part of the necessary equipment of every American citizen.

The Textbook. The makers of textbooks have coöperated in the movement to aid the progress of mathematics. The large majority of textbooks have been prepared solely with a view to assist in the improvement of instruction in our elementary and secondary schools. Much attention is paid to the needs and interests of the children especially in matters of type, proper spacing, and the like. In this respect our books excel those of foreign countries, but theirs are generally more scholarly.

Professor David Eugene Smith's work in the history and background of mathematics has been one of the important factors in improving the general form of the texts. This is seen in the use of pictures, in better diagrams, in historical information, and in reproductions of pages from interesting old books.

Junior high school textbooks show more progress in their willingness to surrender the logical divisions for the sake of a better learning order than do those of the senior high school. The former are free to benefit by this more psychological development and the traditional barriers of arithmetic, algebra, trigonometry, and intuitive geometry are not so impenetrable as those between demonstrative geometry and the other subjects. However that may be, the fact remains that several series of books have been labeled merely Mathematics I, II, and III, and have attempted with some degree of success to organize the material on the basis of learning difficulty rather than on the nature of the material, and have largely disregarded the former compartment method of learning.

Proposed Course in Junior High School Mathematics. The result of the combined efforts of all those interested in the teaching of junior high school mathematics has had a salutary effect. We have to-day a rather wide agreement as to the general features of

in Mathematics for the Eleventh and Twelfth School Years," *The Mathematics Teacher,* 19 :1-8.

[58] See Schorling and Reeve, *General Mathematics,* Book I, Chap. 10. Ginn and Co., 1919. See also Roe, H. B., Smith, David Eugene, and Reeve, W. D., *Mathematics for Agriculture and Elementary Science,* Chap. 6, Ginn and Co., 1928.

the course in junior high school mathematics, even though the order of treatment of topics is not standardized as is the case in the senior high school. Thus, it is quite generally agreed now that the course in mathematics in the junior high school should be determined by the general purpose in teaching any subject, namely, to develop well-educated citizens.

This difference in purpose makes a difference in content possible. It permits us to open the door of mathematics to every boy and girl so as to give a broad view of the subject in order that each one can choose according to his ability and preference, having seen the general nature of the subject and what the science means. If his taste and needs require it, the pupil should be permitted and encouraged to go on. If not, he should not be forced to continue the study too long.

Because of the wide range of individual differences in native ability, experience, and interests we need to keep in mind the guiding principle in the selection of subject matter for each grade. *This principle states that the subject matter selected should be that material which will be most valuable to the pupil, provided he leaves school at the end of that year.*

Seventh Grade. In accordance with this principle the aim in the seventh grade should be to keep up a proper use of the fundamental skills in arithmetic which the pupil has learned in the first six grades. This is done by giving such applications in the arithmetic of the home, of the store, of the bank, of thrift, and the like as the well-educated citizen is likely to need. In addition, the pupil is introduced to the study of intuitive geometry, a subject in which he looks at a figure and draws certain conclusions. For example, he looks at an isosceles triangle and says "the base angles are equal" because he cannot conceive of their being otherwise. In other words, "he feels it in his bones."

There are three questions in geometry which may be asked about an object. First, "Where is it?" Second, "What is its shape?" Third, "What is its size?" The answers to these questions give rise to the geometry of position, form, and size. These topics should be treated in such a way as to give the pupil some idea of geometric forms in nature, architecture, design, and the like. Under the head of the geometry of size the pupil is introduced to simple problems of direct measurement and thus is led to understand simple algebraic formulas like $C = \pi d$, $A = \pi r^2$, $d = rt$, and the like. The need for

a knowledge of evaluation of these simple formulas leads to such equations as $3l = 12$, $n + 4 = 20$, $C - 5 = 14$, and $\frac{1}{2}b = 6$, whose solutions can easily be presented in the seventh grade.

Eighth Grade. In the eighth grade a little more than one-half of the pupil's time should be given to a further consideration of intuitive geometry and to the fundamental applications of arithmetic, such as problems of trade, banking, insurance, corporations, and the like. The remainder of the year should be given over to a study of algebra. It is better to continue the study of algebra which was begun in the seventh grade and to finish it in the ninth year than to condense all the treatment in the ninth grade.

If a pupil ever uses algebra at all, his greatest need will be a knowledge of the formula. Here is where he gets his idea of what algebra means. Even if he doesn't use formulas, he must read them. And so it is with the statistical and the mathematical graph. Moreover, if the pupil uses formulas at all, he will need to know how to solve the simpler types of equations. To these three important ideas we may add directed numbers as the fourth thing in algebra which the well-educated citizen should know.

Ninth Grade. In the ninth grade the course in elementary algebra should be completed and a unit of numerical trigonometry given. There is no question that intuitive geometry correlates well with both algebra and trigonometry. This can easily be done because the trigonometry is based on intuitive geometry and the pupil's previous knowledge of algebra. Moreover, there is to-day a clearer conception than formerly of the nature of trigonometry and its relation to modern life. It gives the pupil a knowledge of indirect measurement in contrast to direct measurement where the measuring instrument is laid directly on the distance to be measured. It is much easier than many of the traditional algebraic topics, is far more important, and is more interesting to the pupils.

A short course in demonstrative geometry should be given in the ninth grade so that the pupil will have a chance to find out what it means to prove something. This can be done with a few axioms, postulates, and theorems reinforced by some carefully planned work on original exercises. It is my belief that it is better to spread this work out over the year, but in any case opinion seems to be increasingly in favor of the idea of some kind of geometry course in the ninth grade for those who will not continue in school.

It is clear that the junior high school course briefly described

above is general mathematics in the best sense. With this basis it is possible in the tenth, eleventh, and twelfth years to give a more enriched course to those pupils who are able and interested enough to study further.

Proposed Course in Senior High School Mathematics. The purpose of the mathematics course in the senior high school is to meet the needs of the following four groups of pupils:

1. Those who intend to go on to colleges and technical schools.
2. Those who are going to specialize in commercial work that requires mathematics, especially algebra.
3. Those who expect to specialize in science.
4. Those who desire to study mathematics further because they like it.

Because of the purpose outlined above it would seem that the senior high school course in mathematics should be made elective.

Probable Alternative Courses. In addition to the probable foundation courses in algebra, plane and solid geometry, and plane and spherical trigonometry traditionally given in the ordinary four-year high school, certain other alternative courses may be mentioned.

1. *College Algebra.* In an increasing number of schools there is offered in the senior year a half-year course in college algebra. This would be one profitable way to spend the time if such a course included what is essentially college algebra. In too many places, however, such a course is college algebra only in name, being mostly reviews of the more difficult parts of elementary and intermediate algebra with only an occasional glimpse at new material on the college level. One solution for such a situation is the complete adoption of the general mathematics program all through the senior high school—a plan which will presently be discussed.

2. *Courses for Future Mathematicians.* For the future mathematicians there will probably be offered certain half-year courses that will be largely informational and basic. Chief among these are college algebra, analytic geometry, projective geometry, the calculus, and possibly a course in what we might call college geometry—geometry to correspond with the course called college algebra.

The decision as to which of these courses is to be offered will doubtless be left to the particular wishes of the instructor who will have to teach the subject.

3. *Courses for Vocations.* There will also be offered in some of our schools courses for those who are preparing to enter vocations. Thus, we may find a course in mechanics for those so inclined, a course in commercial arithmetic or algebra, and possibly a course in machine shop mathematics.

Work of the Tenth Grade. The central feature of this year is the fundamental elements of plane and solid geometry. If a pupil has had the general mathematics course of the ninth grade described above, he is familiar with a large number of definitions and concepts that will form an excellent basis for the work of the tenth grade. Algebra and trigonometry should be used wherever they help to clarify the proofs. Trigonometry correlates well with intuitive geometry and with algebra, but not with demonstrative geometry. To the extent to which it is permissible to introduce mensuration, just so far is it feasible to introduce trigonometry into demonstrative geometry. Both mensuration and trigonometry may have a place at this time for the purpose of changing the emphasis and of providing the pupils with a mental rest from the pursuit of logical demonstration.

Work of the Eleventh Grade. For those who continue the study of general mathematics through the eleventh grade the course should be built up around algebra, trigonometry, and the simpler elements of analytic geometry as the central features. Some teachers may prefer to introduce the calculus at this time or even earlier. There is no reason why this should not be done in certain schools. The following outline for the work of this grade is meant to be merely suggestive of the general field and not of the order of presentation:

I. Dependence.
   1. Meaning of dependence,—functional relationship.
   2. Review and extension of the earlier work on algebraic functions.
   3. Trigonometric functions as examples of transcendental functions.
   4. Functional notation.
   5. Evaluation of functions.
   6. Determination of functions.
      a. From empirical data and tables.
      b. Empirical *vs.* arbitrary functions.
   7. Classification of algebraic functions.
   8. Graphs of algebraic functions.
      a. Variation of function.
      b. Graphical introduction to maxima and minima.

9. Fundamentals of statistical method.
   a. Theory of measurement.
   b. Fundamental ideas.
   c. Applications.

II. Trigonometric Functions of Any Angle.
1. Review and extension of the earlier work.
   a. Ratio definitions, including secant, cosecant, and cotangent.
   b. The right triangle where the acute angle is 30°, 45°, or 60°.
   c. Similarity of triangles. No trigonometric function of a given acute angle has more than one value.
   d. Given the value of a trigonometric function of an acute angle A, to construct A and determine the other trigonometric functions.
2. Changes in the functions of A as A increases from 0° to 360°. Graphical introduction.
3. Positive and negative angles of any size.
   a. Angles whose initial sides and radius vectors are respectively identical.
   b. The signs (quality) of the functions of angles in the various quadrants.
4. Functions of $(-A)$ in terms of A.
5. Functions of $n \times 90° \pm A$, $n \times 180° \pm A$, and $n \times 360° \pm A$.
6. Line definitions of the trigonometric functions.
   a. Graphical representation.
   b. Application to wave motion, electricity, and the like.
7. Table of natural functions.
8. Applied problems.
9. Fundamental trigonometric relations like

$$\sin x = \frac{1}{\csc x} \text{ and } \tan x = \frac{\sin x}{\cos x}.$$

   a. Proving identities.
   b. Applications to physics, navigation, and surveying.

III. Logarithms and Applications.

IV. Solution of Triangles.
1. Right triangles.
2. Oblique triangles.
3. Applications.

V. Identities.
1. In elementary operations.
   a. Work on functions extended.
   b. Use of analytic method, especially in proportion.
   c. Factoring.
      (1) Review and extension of earlier types.
      (2) Factor theorem.
      (3) Factors of $x^n \pm y^n$.
      (4) Mathematical induction.

2. In formulas.
   a. Review and extension of earlier work.
   b. Addition and subtraction theorems of trigonometry.
   c. Trigonometric functions of $2A$ and $\frac{A}{2}$.
   d. Applications of *b* and *c* above.
      (1) In evaluating certain functions.
         For example, $\sin 75° = \sin (45° + 30°) = ?$
      (2) In proving certain identities.

VI. Straight-Line Formulas.
   1. Distance formula.
   2. Mid-point formula.
   3. Slope of a line, parallel lines, a perpendicular.
   4. Point-slope form.
   5. Slope-intercept form.
   6. Two-point form.
   7. Two-intercept form.
   8. Distance from a point to a line.

VII. Equations.
   1. Linear.
   2. Quadratic.
      a. In one unknown.
         (1) Maxima and minima.
         (2) Radical equations.
      b. In two unknowns.
      c. Theory of quadratic equations.
   3. Trigonometric.
      a. Radian measure.
      b. Two acute angles are equal if any trigonometric function of the one is equal to the same function of the other. Explanation of how the value of an angle is obtained and thus how to solve a trigonometric equation.
      c. Solution of trigonometric equations.

VIII. Series.
   1. As an example of an identity.
   2. General nature.
   3. Classes to be considered.
      a. Binomial Theorem.
      b. Arithmetic.
      c. Geometric.
   4. Applications.

IX. General Number System.
   1. Integers.
   2. Fractions.
   3. Negative.
   4. Surd.
   5. Simple imaginaries.

Work of the Twelfth Grade. There is no reason why pupils who wish to elect general mathematics in the twelfth grade should not be provided with a suitable course. There is probably no best course, and a great deal of experimental investigation should be carried on. The following outline is offered as a suggestion for such a course:

I. Variation of Functions.

1. Graphical representation of functions, to bring out variation of different kinds.
   - *a.* Mechanical graphs like Weather Bureau records.
   - *b.* Review of details in graphic work, scales, and the like.
2. Mathematical method of studying functions.
   - *a.* Who makes such study? Illustration.
   - *b.* Why such study is not made of all existing functions.
3. Review of functional notation.
4. Directed lines. Rectangular coördinates.
   - *a.* Intercepts.
   - *b.* Infinitely large or infinitely small functions.
5. What is meant by a rate.
   - *a.* Rate of change of a function.
   - *b.* Uniform rate of change of a linear function.
     - (1) Slope of the straight line $y = mx + b$.
     - (2) Uniform acceleration.
     - (3) Applications.
   - *c.* Average rate of change of a function.
   - *d.* Instantaneous rate of change of a function.
   - *e.* Graphic methods involving *a, b, c,* and *d.*
   - *f.* Small intervals.
   - *g.* Interpolation by proportional parts.
6. Important problems.
   - *a.* Mean-value.
     - (1) Average value throughout a certain interval.
     - (2) Applications.
       - (*a*) To find the distance traveled.
       - (*b*) In physics.
       - (*c*) In geometry.
   - *b.* Extreme-value.
     - (1) Review and extension of maxima and minima.
     - (2) Applications.
   - *c.* Zero-value.
     - (1) Review of earlier equation work with emphasis on the idea of variation.
     - (2) Review of graphic methods of solving simple equations and extension to the solution of higher equations.

(3) Theory of equations.
(*a*) Number of roots.
(*b*) Location principle.
7. Deriving formulas.

II. The Notion of a Limit.
1. Elementary notion.
a. Numerical.
b. Geometric.
2. Relation to instantaneous velocity.
3. Instantaneous direction.
a. What a tangent line is.
b. Slope.
4. Explanation of limit.
5. Explanation of notation used.

III. Differentiation and Its Applications.
1. The idea of a derivative of a function.
2. Meaning of $\frac{dy}{dx}$.
3. Differentiation of simple functions, formulas.
4. Maxima and minima.

IV. Integration and Its Applications.
1. Integration explained.
2. Formula for integration.
3. Notation.
4. Applications to areas, surface, volumes.
5. Other applications.

V. Commercial Algebra.
1. Review.
a. Exponential and logarithmic functions.
b. Growth curves.
c. Compound interest.
d. Annuities.
e. Business depreciation.
f. Differentiation and integration.
2. Series.
a. Evaluation of functions.
b. Mathematics of investment.
(1) Accumulation.
(2) Life insurance.

VI. Permutations, Combinations, and Probabilities.
1. Permutations.
2. Combinations.
3. Chance explained.
4. Simple and compound probability.
5. Normal distributions, surfaces, and curves.
6. Errors of measurement.

7. Applications.
    a. Life insurance problems.
    b. Mendelian theory of inheritance.

VII. The Number System Extended.
1. Review and extension of the real number system.
2. Similarly for imaginary numbers.
3. What a complex number is.
4. Polar form of a complex number. Polar coördinates.
5. Applications.

III. Methods of Instruction

Administrative Trends. By administrative trends we mean the external schemes of organization whereby teaching and learning are accomplished. It will be worthwhile to consider some of the most outstanding of these trends and see if we can discover what progress has been made.

The Laboratory Method. Among the remedies suggested is the laboratory method already referred to, where the instructor is supposed to follow something of the order of procedure commonly employed in the physical sciences. The teacher meets his pupils part of the time for purposes of discussion or "recitation" and devotes the other periods to study and investigation.[59]

Individual Differences and Needs. We are now giving serious consideration to the variation in the abilities and needs of individual pupils. This factor alone probably constitutes the greatest problem in the American secondary school to-day. Through tests of one kind or another we are now able to classify children into ability groups; and yet in many schools we go on with our teaching as though no differences exist. To continue such a practice is unwise. In a democracy, if anywhere, leadership is essential and we are not developing our leaders. Moreover, we are not doing much to develop intelligent followership in those who cannot lead. The study of needs brought more definitely to light by a program of testing has introduced into the teaching of mathematics grouping according to ability, enriched curricula, and opportunity classes.

Homogeneous Classification. Perhaps one of the biggest changes is the practice of grouping by ability or homogeneous classification. This practice enables us to let each class advance at

[59] Newhall, Charles W., "The Teaching of Algebra by the Laboratory Method," *School Science and Mathematics*, 5:40-45. See also Jones, Franklin T., "Some Experiences in Laboratory Mathematics and Their Results," *School Science and Mathematics*, 5: 406-10.

its own rate and allows us to enrich the course beyond its minimum essentials for those of high ability. This is possible because only those who desire and who have the ability to continue mathematics are urged to do so. The modern twelfth-year group in mathematics is even more selected than that of the traditional type, where there was little preparatory work to make for real understanding of the mathematical processes involved. In connection with ability grouping we use prognostic and diagnostic tests which help us to classify the pupils, and practice or drill tests which enable us to tell where the individual pupil is having difficulty.

Fads in Teaching. It was in 1913 that Professor Dewey's *Interest and Effort in Education* made its appearance. By 1915 "motivation" was the watchword in education. "Vitalizing" the curriculum was the key to success at this time.

Thus, a number of fads in teaching have passed over the country from time to time. There was the laboratory method referred to above, especially in geometry, where the pupil discovered for himself the various truths to be learned. Then came the socialized recitation, supervised study, the project method of introducing and developing subjects,[60] the Dalton system, and the Winnetka plan, each with its own advantages but no one of them meeting satisfactorily all situations. Supervised study, for example, is either study or it is not study. If the pupil does the studying, it may be helpful; if the teacher does it, the time is wasted. If one had control over the fourth dimension, and could travel fast enough, so that he could see into ten thousand classrooms to-morrow, he would find in the vast majority of them no unique plan of teaching in use. No doubt he would find each teacher doing the thing most evidently suited to the situation—sometimes a lecture, other times board work, again seat work or perhaps class discussion with an occasional period given over to diagnostic testing, and the like, Often to-day the pupils select the procedure by which they will learn most easily and most economically. The best teachers do not confine themselves to one method at any time. If they did, the monotony would be unendurable. They try all of these schemes sooner or later and in the end retain what is good in each.

New Spirit in Presenting Subject Matter. In the past twelve years a new spirit has characterized the presentation of subject mat-

[60] Jablonower, Joseph, "The Project Method and the Socialized Recitation," *The Mathematics Teacher,* 21 : 431-41.

ter, permitting a pupil to live his life naturally, with a minimum of restraint and without tasks that are unduly irksome. Thus he is allowed to develop his interest in mathematics largely by his own spirit of curiosity. Moreover, he is directed as in a game;—not driven, not even led, but stimulated by the teacher to discover truths. Professor Dewey once said, "It is pretty generally conceded that the proper method of teaching is to present the facts and let them be worked up according to the capacity of the minds that work upon them."

Accordingly, the emphasis has been shifted from subject matter which puzzles the pupils to the pupils himself. As a result of the breaking down of lines of cleavage, of the bringing down of new material, and of the shift of major consideration from the subject to the child, there has followed a more vital, teachable, and learnable organization of subject matter, both in general courses and in those having traditional labels.

The Testing Program. The testing movement, which has swept the country since the World War is at the present time probably having a greater influence upon the teaching of mathematics than anything else. Attention to individual needs and differences has made homogeneous grouping possible. Scientific construction of curricula in mathematics has been in the foreground during the past two years. Diagnostic testing at the present time promises to revolutionize the teaching of mathematics. Psychological analysis of methods and abilities is doing much to clear up questions of procedure. However, these latter developments are too near at hand for us to get the proper perspective in order to evaluate them.

Uses of Tests. About the year 1910 an attempt was made on a large scale to measure the degree of skill attained by pupils in studying the fundamental processes in arithmetic. Tests were soon used as a basis of comparison of achievement of classes and of schools, a practice which has not met with general approval. Later tests were used as a means of determining whether the material taught was too difficult for the majority of the pupils. They were also used to detect weaknesses in teaching so that errors in the use of principles could be corrected.

In an effort to secure more successful work in mathematics, educators have analyzed the major operations and difficulties into those of a minor nature. The teacher, in order to be successful, must be aware of all the difficulties that will confront the pupil and

prepare him for them. Tests have been prepared which reveal not only ability as a whole in a given operation but detailed abilities as well.

Other New-Type Tests. Nobody will claim, of course, that the new-type tests which are not standardized will be a panacea for all of our testing ills. They also have their shortcomings, as we shall see. As Professor Horn points out, we need to emphasize the method of testing instead of any one test itself. He lists [61] the important purposes of educational tests to which I have made certain additions and amplifications as follows:

1. To give in a brief period of time a rapid survey of all the skills or abilities to be tested. Pupils' difficulties with given topics are often due to neglect of the teacher to provide instruction in certain small details. What is mathematically to the teacher a single skill may be psychologically to the pupil a complex of skills.

The new-type test on the quadratic formula on page 178 will bring out this point in a practical way.

The old essay-type of examination tested only a few skills, and it took a long time to do even that. The standardized test in mathematics has progressed little in this direction, although it has made a start.

2. To remove the personal equation in marking papers; in other words, to make the scoring of tests objective. The marking of examination papers in the past has been highly subjective. This procedure is human. Every teacher is prejudiced with respect to his pupils. Objective tests can be marked in only one way and the record is impersonal.

As in the preceding case, the test on page 179 referring to optional historical information that might be given in certain classes, can be scored in only one way.

3. To show the pupil how efficient he has been. The old essay-type of examination did not do this. It was too narrow in scope to show much of the pupil's mastery of the field.

4. To show the teacher how efficient he has been. The essay-type of examination can reveal how well the teacher has taught only a few things. The modern educational tests through their diagnostic features set forth clearly the strengths and weaknesses of pupils so that the teacher can do remedial work if necessary.

[61] Horn, Ernest, General Preface to *Improvement of the Written Examination*, by G. M. Ruch, Scott, Foresman and Co. 1924.

The Quadratic Formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Taking $ax^2 + bx + c = 0$ as the type form of the general quadratic equation which has the two roots x_1 and x_2, insert the proper values in the following table, using the numbered columns which correspond with the eight given equations, as shown for the first five in Ex. 1:

1. $x^2+3x+2=0$.
2. $x^2-3x+2=0$.
3. $x^2+9x-36=0$.
4. $x^2-13x-40=0$.
5. $x^2-8x=-15$.
6. $x^2+10x=-24$.
7. $x^2-10x=11$.
8. $x^2+14x=32$.

	1.	2.	3.	4.	5.	6.	7.	8.
$a =$	1							
$b =$	3							
$c =$	2							
$-b =$	-3							
$b^2 =$	9							
$4ac =$								
$-4ac =$								
$2a =$								
$b^2 - 4ac =$								
$+\sqrt{b^2 - 4ac} =$								
$-\sqrt{b^2 - 4ac} =$								
$-b + \sqrt{b^2 - 4ac} =$								
$-b - \sqrt{b^2 - 4ac} =$								
$x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} =$								
$x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} =$								

Optional Historical Information

In the left-hand column is the name of some person, place, time, phrase, or people corresponding to some item in the right-hand column. Insert on the dotted lines the letter of that item which corresponds to the numbered entry in the left-hand column:

1. Thales.	.... *a.*	Square on the hypotenuse.
2. Pythagoras.	.... *b.*	First printed edition of Euclid's geometry.
3. Euclid.	.... *c.*	Great Egyptian surveyor. Formula $A = \sqrt{s(s-a)\ (s-b)\ (s-c)}$.
4. Greeks.	.... *d.*	Laid the foundation for geometric proofs.
5. Romans.	.... *e.*	Foundations of demonstrative geometry.
6. Plato.	.... *f.*	Euclid lived about this time at Alexandria.
7. Heron.	.... *g.*	About 1600 B. C. copied an earlier manuscript on Egyptian mathematics.
8. Archimedes.	.... *h.*	Wrote the greatest of the Greek textbooks on geometry.
9. Alexandria.	.... *i.*	To them is due the best work in geometry in early times.
10. Ahmes.	.... *j.*	Plato lived in Athens about this time.
11. Arabs.	.... *k.*	A Greek city at the mouth of the Nile. Euclid lived there.
12. 300 B. C.	.... *l.*	Applied mathematics to practical uses, as in great engineering works.
13. 500 B. C.	.... *m.*	Translated Greek works on geometry, whence they reached Europe again through the Latin.
14. 400 B. C.	.... *n.*	Pythagoras lived in Croton, Italy, about this time.
15. 1482.	.... *o.*	Lived in Syracuse, Sicily. Computed a fairly good value of π.
16. Pons asinorum.	.... *p.*	First English translation of Euclid's great geometry.
17. Golden section.	.... *q.*	Badge of the Pythagorean brotherhood.
18. Pentagonal star.	.... *r.*	The best known of the early translations of Euclid into Latin, about 1260.
19. Campanus.	.... *s.*	Angles opposite the equal sides of an isoceles triangle.
20. 1570.	.... *t.*	Dividing a line into extreme and mean ratio.

This is the strongest feature of such tests. When properly prepared, they reveal the pupil's knowledge and abilities with respect to every essential feature of the work.

5. To measure the value of a given text or method of teaching. The tendency to-day is to build up a course of study and then provide tests to determine how well the objectives are being realized.

6. To find out how long it takes to teach a topic. To-day we do not know how long it takes to teach anything. We know that we are getting mastery on only a few things. With proper testing, for example, we could find out how long it takes to teach a normal group of ninth-grade pupils to factor the difference of two squares to any desired degree of mastery.

7. To serve as teaching devices. No textbook can contain all the exercise material required by modern courses of study, particularly in the fields of oral and of rapid written work. The tests supplement the text and relieve the teacher of the necessity of supplying the additional material.

8. To find out what is a desirable content. No matter how desirable some of our objectives are, it may be that some of them are not within the reach of the pupils we are teaching. It is only by intelligent testing that we can properly decide finally which objectives are valid.

9. To enable a pupil to rate himself on his performance in relation to his former record or that of his fellows. The first of the two schemes is probably the better to use. Just as a man is eager to beat his previous score in golf, so a pupil is usually eager to improve his score in mathematics.

10. To survey the status of teaching in a school system. These tests are designed to afford a comprehensive survey of the work of a semester or a year. They are often given by some one who is surveying the school system and, although general, are useful in indicating the status of teaching in the system.

Present Weakness of New-Type Tests. We should not conclude without pointing out some of the weaknesses of new-type tests.

1. Many of these tests contain obsolete material of no possible importance.

2. Some of the tests are poorly arranged and badly printed.

3. Some tests, especially those made by busy teachers, have had all sorts of impracticable features, such as loose detached sheets,

transparent paper, and the like, that make them unfitted for classroom use.

4. Certain speed tests tend to glorify the machinery of mathematics. Habits on this mechanical part of algebra may be overdone. Skill in algebra must not be obtained at the expense of understanding. There is no value in finding out how fast errors can be made.

5. Only of late has anything been done to test the field of algebra, and only recently has anything of importance been done in geometry, although a good start has now been made.

6. The tests do not measure attitudes, appreciation, and the like. This need not remain true, and at the present time steps are being taken to remedy this defect.

Conclusion. Finally, it is not contended that the new-type tests should replace the more traditional types, but it is suggested that a broader and wiser use be made of these newer instruments of measurement in supplementing the older ones. There is little doubt that the pupils themselves like the new types much better than the older ones. We know from actual use that larger areas of subject matter may be tested in less time by the new-type tests and that the drudgery of scoring is greatly reduced by their use. We have reason to believe that we obtain more information about the extent and quality of a pupil's learning through the use of the newer tests and that our remedial instruction is more intelligent and worthwhile.

In spite of the frequent inadequacy and inaccuracy of teachers' judgments both in setting good examinations and in marking them fairly, it should be more generally recognized that these same teachers are in the long run the ones best situated to do the task. Teachers cannot only learn how to make objective tests that will have both measuring and diagnostic value, but they can also learn to use them intelligently. This ability to use the tests will increase in proportion to the progress that teachers make in understanding more scientific methods of measurement.

IV. Present Indications of Future Progress

Increasing Prestige and Value of Mathematics. Professor Hedrick says that the mathematics developed since 1900 is more important and far-reaching than all that was developed prior to that time. Einstein's Theory, which is really more mathematical

than physical, is as important as the development of the calculus. A vast amount of new mathematics has developed during the Twentieth Century. Eighty per cent more of such material appeared in the journals of mathematics in the United States in 1927 than in 1920.

During the period from 1923 to 1927, mathematicians were awarded important prizes for the most outstanding papers in the entire field of pure science. The winners were Professor Birkhoff of Harvard and Professor Dickson of Chicago. The awards are all the more significant when one realizes that no mathematicians were on the award committee.

Many other instances could be pointed out to illustrate the dynamic nature of mathematical research. Professor Slaught in an article on "Mathematics and Sunshine" [62] says:

> It is well known that one of the major interests of the *Rockefeller Foundation* is the promotion of public health through scientific research in the fields underlying medicine. Their procedure is to delegate to the *National Research Council* the selection of highly trained men whose powers of research in these fields have already been tested and to award them cash fellowships as an incentive to still further prosecute their investigations. The biological sciences, of course, were chosen initially for these fellowship awards, then chemistry as underlying biology, and physics as underlying chemistry, and finally *mathematics* as underlying all the rest.

The following quotation [63] from an editorial in the *Saturday Evening Post* is apropos here:

> Many a bright and promising college man drops his studies along with his athletics. After a few years he takes on weight and becomes heavy on his feet. His intimates make teasing remarks about bay windows; but none will have the hardihood to hint that he has likewise developed a bay window of the mind or has allowed his mental machinery to rust and jam through sheer neglect and shiftlessness. Faithful are the wounds of a friend, but friendship is the price of inflicting them.
>
> Hopeless cases of fine minds gone soft and flabby are so common that it is not too much to say that arrested intellectual development is the great national disease of our educationally privileged classes. Sheer lack of will power and mental stamina makes it difficult or impossible for us to forego ease and rest and attack irksome tasks such as reading the books that harden the brain, but which are so new and strange that they must be studied as a child studies geometry—painfully and doggedly. Since men now in their fifties went to college the whole universe has been taken down and reassembled in a

[62] Slaught, H. E., "Mathematics and Sunshine," *The Mathematics Teacher*, 21 : 249.

[63] Mathematics News Letter, 3 : 4-5.

new and unfamiliar form. Literature, relatively speaking, has been marking time. Science has been going ahead by running leaps. Unfortunately for the casual and easily daunted reader, modern science is written in the language of mathematics and in the dialect of calculus; not only physics, chemistry, and electricity but physiology and the other life sciences. Lack of easy familiarity with higher mathematics is a formidable obstacle between our ignorance and any real grasp of the modern conceptions of the universe we live in; and that obstacle will continue to bar our paths until the extraordinary importance of mathematical studies receives full and practical recognition.

Unrest a Sign of Progress. Since 1910 a spirit of unrest has permeated the field of mathematics. As a rule, this is an indication of progress. The great changes in industry, modes of travel, transportation, and the complex social life in which we live call for a new application of mathematics to meet the problems of everyday life. In fact, as we become better acquainted with the world about us we are forced to think more and more in mathematical terms. All real progress in economics, in business, in industry, in the sciences, and in many other fields depends upon the understanding which the leaders of those fields have of mathematics.

Organizations of Teachers of Mathematics. The united effort of a homogeneous group of any sort has always proved to be one of the best methods of accomplishing great tasks. Many organizations of mathematics teachers have been formed since 1910 not only to advance group interests but to advance as well those of the individual teacher. Witness the report of the National Committee on *The Reorganization of Mathematics in Secondary Education.* The membership of teachers in mathematics clubs [64] and organizations, the magazines, pamphlets, and yearbooks are sufficient evidence that mathematics teachers are awake to existing conditions. Present indications are that mathematics is still to remain a major subject in the secondary schools of the United States.

However, we need to have more frequent meetings of teachers in the various mathematics departments in this country to talk over their common problems. Teachers located in cities should form clubs like those in Buffalo, Chicago, Cleveland, Columbus (Ohio), Detroit, Minneapolis, Philadelphia, New York, St. Louis, and St. Paul. These clubs have regular meetings, and some of them have

[64] See Gugle, Marie, "How Mathematical Clubs and Associations May Become Affiliated with the National Council of Teachers of Mathematics," *The Mathematics Teacher,* 21 : 422-26.

already become branches of the National Council of Teachers of Mathematics. The various state organizations should also be affiliated with the National Council so that the work of improving instruction in mathematics may be done more intelligently.

Professional Advancement. Some teachers of mathematics, like those of other subjects, "get in a rut." They do not read the current educational and mathematical journals. Nevertheless, they are coming more and more to renew their interest in their subject by attendance at summer sessions or at regular sessions during the academic year in reputable institutions of learning. Every teacher of mathematics in this country should be a member of the National Council of Teachers of Mathematics and should read regularly its official organ, the *Mathematics Teacher.* This magazine is the only one in the country devoted entirely to mathematics in the elementary and secondary fields.

The National Council of Teachers of Mathematics. The organization of the National Council of Teachers of Mathematics [65] at Cleveland in 1920 inaugurated a nation-wide program for improving mathematics. Prior to this, as Professor Slaught pointed out at the ninth annual meeting of the Council in Boston in February, 1928, we lacked three essential characteristics for success as a national organization; namely, group consciousness, group pride, and group enthusiasm. With our rapidly increasing membership (from 3,000 in 1927 to more than 5,000 at the present time), with the awakened interest in the yearbooks and the formation of branches of the Council [66] all over the country, we shall soon acquire the three important characteristics referred to above. As Mr. Austin [67] put it at the Boston meeting:

> Counting the meeting for organization, eight annual meetings have been held—two at Cleveland, two at Chicago, one each at Atlantic City, Washington, Cincinnati, and Dallas. Considering the great distances to travel, the attendance has been exceptionally large—averaging about one hundred and seventy-five persons. Ten to twenty states were represented at each meeting. These national meetings have been valuable not only for carrying on the work of the National Council, but also because they have been a great inspiration to the local teachers of the community. The meeting at Dallas, Texas, in 1927

[65] See Austin, C. M., "Historical Account of Origin and Growth of the National Council of Teachers of Mathematics," *Mathematics Teacher,* 21 : 204-13.

[66] Gugle, Marie, "How Mathematical Clubs and Associations May Become Affiliated with the National Council of Teachers of Mathematics, *Mathematics Teacher,* 21 : 422-26.

[67] Austin, C. M., *op. cit.,* p. 121.

well illustrates the point. Two hundred teachers from fourteen states attended. Oklahoma and Texas furnished the largest number. No meeting like this had ever come to that part of the country. The Texas teachers, with one accord, testified to the wonderful impetus and inspiration given to their work by the National Council meeting.

While all the things desired have not yet come to pass, while only a small number of the teachers of mathematics in the United States have been reached and influenced by our organization and its publications, yet the writer of this account feels that the many things already accomplished amply justify the faith of those who founded the National Council and who are still working to give it a larger place in the educational world.

Realization of the Need for Research. Most of the changes that have taken place in the curriculum are due to "external social forces" which have exerted an influence on the schools. Little change has been effected through research. The three or four outstanding examples of recent reforms accomplished as a result of research are given by Professor Judd.[68] He says:

> Ayres changed at a single stroke the content of the course in spelling. Laboratory investigations are directly responsible for the present emphasis on silent reading. Certain studies of the social demands for mathematics have been influential in modifying the amount and kind of mathematics taught in the schools.

Again, Professor Judd says, "If one were disposed to be pessimistic, one would be tempted to use the terms that have frequently been used by critics and would say that fad after fad has been injected into the school program without adequate reason or justification."

More Adequate Preparation of Teachers. Concerning the teacher and his qualifications, the Report of the National Committee states:

> While the greater part of this report concerns itself with the content of courses in mathematics, their organization and the point of view which should govern the instruction, and investigations relating thereto, the National Committee must emphasize strongly that EVEN MORE FUNDAMENTAL is the problem of the teacher—his qualifications and training, his personality, skill and enthusiasm. The greater part of the failure of mathematics is due to poor teaching. GOOD TEACHERS HAVE IN THE PAST SUCCEEDED, AND WILL CONTINUE TO SUCCEED IN ACHIEVING HIGHLY SATISFACTORY RESULTS WITH THE TRADITIONAL MATERIAL; POOR TEACHERS WILL NOT SUCCEED EVEN WITH THE NEWER AND BETTER MATERIAL.

[68] Judd, Chas. H., "The Place of Research in a Program of Curriculum Development," *Journal of Educational Research,* 17 : 313-23.

Schools of education throughout the country are furnishing to ambitious teachers the opportunity to learn more about the solution of their problems by providing summer sessions, extra-mural courses, part-time study, and correspondence courses. A genuine attempt is made to give teachers what they want and need. A significant statement in this connection was made by Professor W. C. Bagley recently when he said that the greatest change in education at Teachers College, Columbia University, during the past ten years has been the increasing emphasis on subject matter.

Outlook for Mathematics. The outlook for mathematics is bright. If the organization of mathematics along better lines is necessary, and improvement in instruction is possible, those best fitted to make the greatest contribution are the mathematics teachers. We should begin by remedying the present defects. The ultimate success of our efforts will be due to the optimism and intelligence we show in admitting the facts and in reorganizing the course accordingly.